South Sudan

Secondary Mathematics 3

Secondary Mathematics has been written and developed by Ministry of General Education and Instruction, Government of South Sudan in conjunction with Subjects experts. This course book provides a fun and practical approach to the subject of mathematics, and at the same time imparting life long skills to the pupils.

The book comprehensively covers the Secondary 3 syllabus as developed by **Ministry of General Education and Instruction.**

Each year comprises of a Student's Book and Teacher's Guide.

The Teacher's Guide provide:

- Full coverage of the national syllabus.
- A strong grounding in the basics of mathematics.
- Clear presentation and explanation of learning points.
- A wide variety of practice exercises, often showing how mathematics can be applied to real-life situations.
- It provides opportunities for collaboration through group work activities.
- Stimulating illustrations.

All the courses in this primary series were developed by the Ministry of General Education and Instruction, Republic of South Sudan. The books have been designed to meet the primary school syllabus, and at the same time equiping the pupils with skills to fit in the modern day global society.

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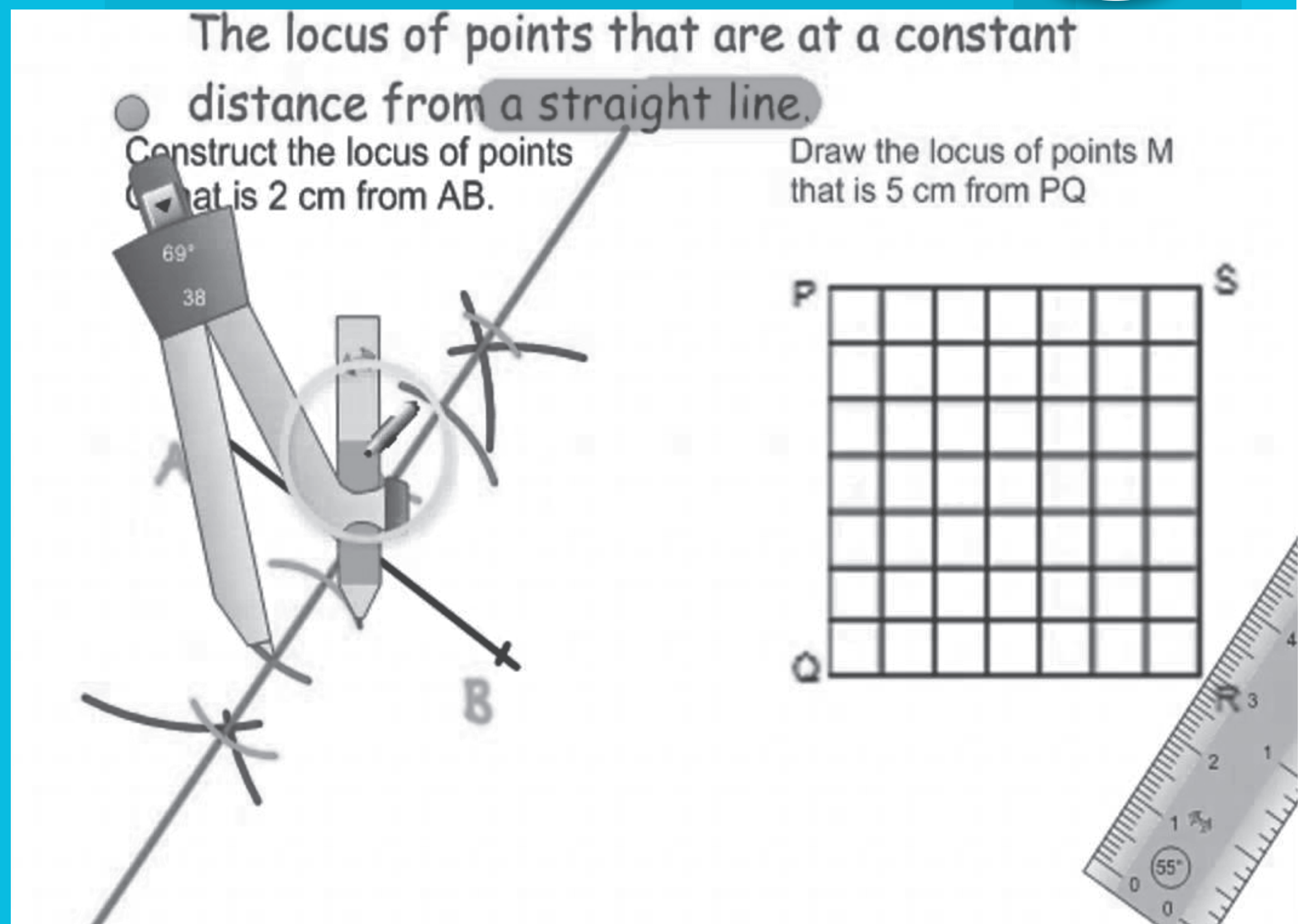
South Sudan

Secondary Mathematics 3

Teacher's Guide



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South Sudan

SECONDARY

3

Mathematics

Teacher's Guide 3

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FOREWORD

I am delighted to present to you this Teacher's Guide, which is developed by the Ministry of General Education and Instruction based on the new South Sudan National Curriculum. The National Curriculum is a learner-centered curriculum that aims to meet the needs and aspirations of the new nation. In particular, it aims to develop (a) Good citizens; (b) successful lifelong learners; (c) creative, active and productive individuals; and (d) Environmentally responsible members of our society. This textbook, like many others, has been designed to contribute to achievement of these noble aims. It has been revised thoroughly by our Subject Panels, is deemed to be fit for the purpose and has been recommended to me for approval. Therefore, I hereby grant my approval. This Teacher's Guide shall be used to facilitate learning for learners in all schools of the Republic of South Sudan, except international schools, with effect from 4th February, 2019.

I am deeply grateful to the staff of the Ministry of General Education and Instruction, especially Mr Michael Lopuke Lotyam Longolio, the Undersecretary of the Ministry, the staff of the Curriculum Development Centre, under the supervision of Mr Omot Okony Olok, the Director General for Quality Assurance and Standards, the Subject Panelists, the Curriculum Foundation (UK), under the able leadership of Dr Brian Male, for providing professional guidance throughout the process of the development of National Curriculum, school textbooks and Teachers' Guides for the Republic of South Sudan since 2013. I wish to thank UNICEF South Sudan for managing the project funded by the Global Partnership in Education so well and funding the development of the National Curriculum, the new textbooks and Teachers' Guides. I am equally grateful for the support provided by Mr Tony Calderbank, the former Country Director of the British Council, South Sudan; Sir Richard Arden, Senior Education Advisor of DfID, South Sudan. I thank Longhorn and Mountain Top publishers in Kenya for working closely with the Ministry, the Subject Panels, UNICEF and the Curriculum Foundation UK to write the new textbooks. Finally, I thank the former Ministers of Education, Hon. Joseph Ukel Abango and Hon. Dr John Gai Nyuot Yoh, for supporting me, in my role as the Undersecretary, to lead the Technical Committee to develop and complete the consultations on the new National Curriculum Framework by 29 November 2013.

The Ministry of General Education and Instruction, Republic of South Sudan, is most grateful to all these key stakeholders for their overwhelming support to the design and development of this historic South Sudan National Curriculum. This historic reform in South Sudan's education system is intended to benefit the people of South Sudan, especially the children and youth and the future generations. It shall enhance the quality of education in the country to promote peace, justice, liberty and prosperity for all. I urge all Teachers to put this textbook to good use.

May God bless South Sudan. May He help our Teachers to inspire, educate and transform the lives of all the children and youth of South Sudan.

Deng Deng Hoc Yai, (Hon.)
Minister of General Education and Instruction, Republic of South Sudan

Table of Contents

Introduction

Teaching Mathematics is taking place in rapidly changing conditions. It is necessary to look for optimal didactic and educational solutions encompassing goals and contents as well as forms and teaching methods allowing for preparing students to face the challenges of the contemporary world.

The most significant role of educational system in terms of teaching Mathematics is developing and promoting subject competences as an important factor fostering student's personal development and the development of society. Well organised mathematical education facilitates logical thinking and expressing ideas, organizing own work, planning and organizing the learning process, collaboration and responsibility; it prepares for life in a modern world and enables to perform many jobs.

The teacher is required to pay more attention to students' awareness of developing learning skills and study habits, recognizing and analysing problems and predicting solutions to them. Undeniably, the implementation of modern teaching methods and techniques enhances students' curiosity about Mathematics and increases their understanding of the basis of mathematical and scientific knowledge. In accordance with the trends teaching Mathematics is supposed to help students understand and solve everyday problems.

The aim of teaching Secondary Mathematics is to encourage contemporary students to work in class, acquire knowledge and skills that are necessary in life. Moreover, research shows that teachers applying active methods assess the effectiveness of their work and how students respond to this way of teaching.

About this guide

The purpose of this guide is to offer suggestions that are helpful to Secondary 3 Mathematics teachers on planning, organizing, executing and evaluating the learning and teaching of mathematics. The suggestions will serve as useful starting points to the teachers who are expected to be dynamic innovative and creative to make the leaning process fit the learners.

The guide is to be used alongside Mathematics Students book for secondary 3. It consists of 4 units, in line with secondary 3 mathematics syllabus.

Each of the unit is structured to contain:

1. Introduction
2. Objectives
3. Teaching/Learning Activities
4. Answers to the exercises given in students secondary 3 book.

In each case, the introduction highlight the relevant work than learners are expected to have covered in their previous mathematics units and what they are expected to have covered previously. It also highlights what they are expected to cover in the unit. The teacher is expected to make a quick link up of previously learnt concepts. Learners should be able to make relevant references to their previous work. Where possible the mathematics teacher makes an entry behavior evaluation as a revision on previously learnt units related to the unit under study.

The unit objectives specify the skills (cognitive, affective, and psychomotor) that teachers will use to enable learners understand each unit. The objectives are likely to serve a useful purpose if they when stated to reflect the local conditions of the learner. For example, the type of students and the available learning resources. The teacher may break down the unit objectives to various objectives that enhance the learners understanding of the process involved and to suit different situations in the lesson, schools, society and the world at large.

Teaching/learning activities highlight the most noticeable and important. Points encountered in the learning process and suitable techniques to be used in handling each objective(s).

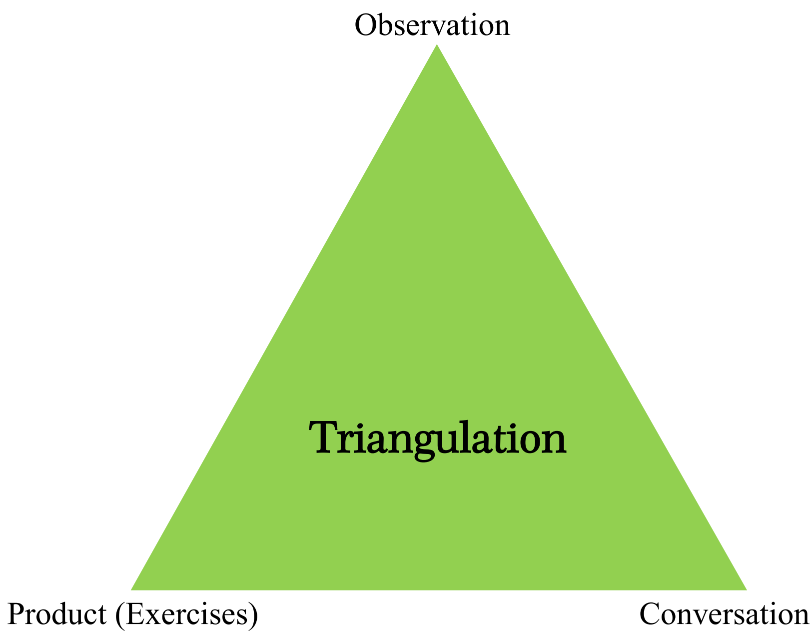
Answers to each exercise in the students' book are provided in these teachers guide. It is contemplated that the most conducive and favorable outcome from the guide will be realized if other sources of learning mathematics are properly organized and used.

Among others, the following should be used alongside the guide:

1. The Schemes of work
2. The teacher's Lessons plans.
3. The Records of the work covered by the learners.

Making Classroom Assessment

- Observation – watching learners as they work to assess the skills learners are developing.
- Conversation – asking questions and talking to learners is good for assessing knowledge and understanding of the learner.
- Product – appraising the learner's work (writing report or finding, mathematics calculation, presentation, drawing diagram, etc.).



To find these opportunities, look at the "Learn About' sections of the syllabus units. These describe the learning that is expected and in doing so they set out a range of opportunities for the three forms of opportunity.

UNIT 1: LOGARITHMS

Maths Secondary 3	**Unit 1: Logarithms**
Learn about	**Key inquiry questions**
Learners should investigate and understand the laws of logarithms and logarithmic equations and apply them in calculations. They should investigate how logarithms are used to ease multiplication and division of large real numbers. They should use logarithmic tables to investigate how logarithms of e. g. 5 and 7 were calculated and solve equations involving logarithms and indices, e.g. $x = a^n$ and $\log x = n$	- What is a logarithm? - What are the laws of logarithm and how do we apply them in calculations? - What is anti-logarithm? - How do we recognise logarithmic equations?

Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**
- Understand logarithms - Know the laws of logarithms and apply in calculation	- Investigate the laws of logarithm and how they are used to simplify calculation of large numbers - Investigate how the log of a number is calculated - Use logarithmic equations to solve problems	- Appreciation of logarithms as a useful and essential tool in calculation - Resilience when doing calculations

Contribution to the competencies:
<u>Critical thinking</u> through analysis and investigation <u>Co-operation and communication</u> through teamwork
Links to other subjects:

What is a logarithm?

Exercise 1

The teacher should ensure there is understanding of index notation and logarithmic notation.

Q1 and 2: Each group should show understanding by presenting the solutions to one of the questions.

Q 3: Groups should discuss the example. The teacher could ask prompting questions to find out if the students understand the steps.

b) 6 c) 0 d) $\frac{1}{2}$ e) -4 f) $\frac{3}{5}$ g) 1

Q4. Allow students to discuss these questions. They should understand that you cannot find the logarithm of a negative number or of zero.

Laws of logarithm

Exercise 2

1. a) 83 b) 14 c) $2\frac{31}{32}$ d) 9 e) 6

Solving logarithmic Equations

Exercise 3

We add logarithms when we multiply numbers and we subtract logarithms when we divide numbers

4. a) 37.41

   b) 0.01188

   c) 10.9

   d) 0.05

e) 217

f) 6600

Exercise 4: To be done in groups and presented before the whole class

Guide the learners in completing the exercise in groups. Allow enough time for discussions and presentations in class. Learners should be encouraged to compare information from various sources on the subject of logarithms.

During presentations simple and clear explanations and justifications should be used. Encourage students to use proper English and make corrections when mistakes are made.

Encourage other groups to voice their observations on a presentation after the presentation has been made. Those listening should not interrupt the group making its presentations but should instead voice their agreements or disagreements after the presentation.

Application of logarithms

Logarithms put numbers on a human-friendly scale.
Large numbers break our brains. Millions and trillions are "really big". The trick to overcoming "huge number blindness" is to write numbers in terms of "inputs" (i.e. their power base 10). This smaller scale (0 to 100) is much easier to grasp:

- power of 0 = 10^0 = 1 (single item)
- power of 1 = 10^1 = 10
- power of 3 = 10^3 = thousand
- power of 6 = 10^6 = million
- power of 9 = 10^9 = billion
- power of 12 = 10^{12} = trillion
- power of 23 = 10^{23} = number of molecules in a dozen grams of carbon
- power of 80 = 10^{80} = number of molecules in the universe

A 0 to 80 scale took us from a single item to the number of molecules in the universe.

Logarithms count multiplication as steps
Logarithms describe changes in terms of multiplication: in the examples above, each step is 10 times bigger.

When dealing with a series of multiplications, logarithms help "count" them, just like addition counts for us when effects are added.

Order of magnitude
In computers, where everything is counted with bits (1 or 0), each bit has a doubling effect (not 10×). So going from 8 to 16 bits is "8 orders of magnitude" or 2^8= 256 times larger. ("Larger" in this case refers to the amount of memory that can be addressed.) Going from 16 to 32 bits means an extra 16 orders of magnitude, or 2^{16}~ 65,536 times more memory that can be addressed.

Interest Rates
How do we figure out growth rates? A country doesn't intend to grow at 8.56% per year. You look at the GDP one year and the GDP the next, and take the logarithm to find the *implicit* growth rate. Logarithms are how we figure out how fast we're growing.

Measurement Scale: Richter, Decibel, etc.
The idea is to put events which can vary drastically (earthquakes) on a single scale with a small range (typically 1 to 10). Just like PageRank, each 1-point increase is a 10 times improvement in power. The largest human-recorded earthquake was 9.5; the Yucatán Peninsula impact, which likely made the dinosaurs extinct, was 13.

Decibels are similar, though it can be negative. Sounds can go from intensely quiet (pindrop) to extremely loud (airplane) and our brains can process it all. In reality, the sound of an airplane's engine is millions (billions, trillions) of times more powerful than a pindrop, and it's inconvenient to have a scale that goes from 1 to a trillion. Logs keep everything on a reasonable scale.

UNIT 2: MEASUREMENT AND GEOMETRY

Mathematics Secondary 3	**Unit 2: Measurements and Geometry**
Learn about	**Key inquiry questions**
Learners should investigate the use of calculators by estimating and checking the perimeter, surface area and volume of irregular objects and know how to calculate the volume of three-dimensional figures. They should know how to find the area of combined shapes and the circumference and area of a circle. Learners should know that a locus is the path moved by a point according to a rule and investigate some rules. They should know how to prove Pythagoras theorem and understand and use loci and discover that the equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$. Learners should understand significant numbers and their use and explain the use of indices and notations in writing astronomical figures.	- How accurate are we at estimation and approximation? - How do we investigate estimation and approximation of surface area and volume of irregular objects? - How do we find and calculate volume of a three-dimensional solid? - What do you need to know to derive the general equation of a circle?

Learning outcomes

Knowledge and understanding	**Skills**	**Attitudes**
- Understand the mathematical relationships of three dimensional figures - Understand the proof of Pythagoras theorem - Understand and use loci Know and understand the equation of a circle is $(x-a)^2 + (y-b)^2 = r^2$	- Use calculators and performance of calculations - Estimate and approximate surface area and volume of irregular objects - Carry out the methods of proving Pythagoras theorem and drawing the locus of a moving point in the plane - Use the general equation of a circle in solving problems	- Appreciate estimation and approximation in general as a problem solving tool in quick decision making

Contribution to the competencies: <u>Critical thinking</u> through analysis and investigation <u>Co-operation and communication</u> through teamwork
Links to other subjects: Social Studies Science

Accuracy: Approximation

Exercise 1: Rounding

Rounding numbers are important to preserve figures in calculations and to record long numbers. When rounding whole numbers there are two rules to remember.

First, you need to understand the term "rounding digit". When asked to round to the closest ten, your rounding digit is the second number to the left (ten's place) when working with whole numbers. When asked to round to the nearest hundred, the third place from the left is the rounding digit (hundreds place).

Rules for rounding whole numbers

Rule One.

Determine what your rounding digit is and look to the right side of it. If the digit is 0, 1, 2, 3, or 4 do not change the rounding digit. All digits that are on the right-hand side of the requested rounding digit will become 0.

Rule Two.

Determine what your rounding digit is and look to the right of it. If the digit is 5, 6, 7, 8, or 9, your rounding digit rounds up by one number. All digits that are on the right-hand side of the requested rounding digit will become 0.

Rounding rules for decimal numbers

When rounding numbers involving decimals, there are 2 rules to remember:

Rule One

Determine what your rounding digit is and look to the right side of it. If that digit is 4, 3, 2, or 1, simply drop all digits to the right of it.

Rule Two

Determine what your rounding digit is and look to the right side of it. If that digit is 5, 6, 7, 8, or 9 add one to the rounding digit and drop all digits to the right of it.

Rule Three:

This rule provides more accuracy and is sometimes referred to as the 'Banker's Rule'. When the first digit dropped is 5 and there are no digits following or the digits following are zeros, make the preceding digit even (i.e. round off to the nearest even digit).

E.g., 2.315 and 2.325 are both 2.32 when rounded off to the nearest hundredth.

Note: The rationale for the third rule is that approximately half of the time the number will be rounded up and the other half of the time it will be rounded down.

Answers

1.

a) 69 000

b) 74 000

c) 89000

d) 4000

e) 100 000

f) 1 000 000

2.

a) 78 500

b) 6900

c) 14 100

d) 1000

e) 3000

f) 8100

3.

a) 490

b) 690

c) 8870

d) 0

e) 1000

f) 80

Decimal Places

Exercise 2: Decimal places

Facts about decimals

1. Decimals or decimal fractions are another way of writing numbers which are not whole numbers.
2. The decimal point separates the whole number fractional parts or parts of the whole.
3. The decimal point is placed to the right of the unit column and is used to indicate the border line between the whole numbers and numbers less than one. The numbers to the immediate right of the decimal point represent the parts. The role of the decimal point is to indicate the unit position (to its left).
4. Fractions are either finite or infinite repeating decimals. It is possible to change any fraction into a decimal point. However, come fractions like a $\frac{1}{3}$ are never ending (0.333333 with the 3 repeated indefinitely) so are called infinite repeating decimals or recurring decimals.

1.

a) 5.6

b) 0.7

c) 11.9

d) 12.0

e) 1.0 f))1.0

2.

a) 6.47 d) 100.00

b) 9.59 e) 3.00

c) 16.48 f) 9.30

Significant figures

Exercise 3: Significant figures

Guide learners in holding discussions in pairs about significant figures. Walk around the classroom listening in on the discussions and providing direction to the students. After the pair work, a few volunteers can share their answers concerning the activity just concluded. Allow the rest of the students to react to the presentations before tackling exercise 3.

Answers

1.

a) 50 000 d) 15.0

b) 48 600 e) 0.09

c) 2.58 f) 0.0031

Upper and lower bounds

Task

A rectangular card measures 5.3 cm by 2.5 cm. Find

a) the limits within which the length and width lie
b) the area of the card, using the stated values
c) the limits within which the area lies

d) the relative error in the calculation of the area

Solution

a) The dimensions have been measured to the nearest 1 d.p.(0.1), therefore the absolute error is $\frac{1}{2} \times 0.1 = 0.05$

Upper limit: length =5.35cm width=2.55cm

Lower limit: length = 5.25cm width = 2.45cm

b) Area $5.3 \times 2.5 = 13.25cm^2$

c) Maximum area= $\boldsymbol{maximum\ length\ \times maximum\ width}$

$= 5.35 \times 2.55 = \mathbf{13.6425\ cm^2}$

$\boldsymbol{Minimum\ area = minimum\ length\ \times minimum\ width}$

$= 5.25 \times 2.45 = \mathbf{12.8625 cm^2}$

d) Absolute error for area = $\frac{1}{2}(13.6425 - 12.8625) = 0.39$

Therefore $relative\ error = \frac{0.39}{13.25} = \mathbf{0.29(2\ s.f)}$

Exercise 4: Approximation

1. $\frac{0.01}{8} \times 100 = 0.125\%$

2. $\frac{0.05}{492.81} \times 100 = 0.0101\%$

Standard form:

Exercise 5: Use of calculators

Check for students understanding of how to input values when using the calculator.

1. a) 106.64=106 m^2 b) 39.74=39.7m^2 c) 421.04=421 cm^2

2. a) S.A=274.8=275 cm^2 Vol=207.9= 208 cm^3

   b) S.A = 1288=1290 cm^2 Vol=1760cm^3

Equation of a circle

Investigation: Equation of a circle

In general, the equation of a circle center O (0, 0) and radius r is given by the equation

$$\boldsymbol{x^2 + y^2 = r^2}$$

If the center of the circle is $A(a, b)$, then the equation is

$$\boldsymbol{(x - a)^2 + (y - b)^2 = r^2}$$

Exercise 7:

1. $(x + 1)^2 + (y - 5)^2 = 9$

2. $(x + 3)^2 + (y - 3)^2 = 25$

$$x^2 + 6x + 9 + y^2 - 6y + 9 = 25$$

$$x^2 + 6x + y^2 - 6y = 7$$

3. a) $x^2 - 6x + 9 = (x - 3)^2$
   b) $y^2 + 14y + 49 = (y + 7)^2$

   Radius = 6 units, center (3,-7)

4. Radius = 4 units Center (3,-2)

5. Midpoint of AB $\left(\frac{5-1}{2}, \frac{1+1}{2}\right) = (2,1)$

Center (2, 1)

Length of AB $\sqrt{(5+1)^2 + (1-1)^2} = \sqrt{36} = 6$

Diameter = 6 units, **radius = 3 units**

Equation of the circle is $\boldsymbol{(x-2)^2 + y - 1)^2 = 9}$

UNIT 3: ALGEBRA

<table>
<tr><td colspan="2">Math Secondary 3</td><td>Unit 3: Algebra</td></tr>
<tr><td colspan="2">Learn about</td><td>Key inquiry questions</td></tr>
<tr><td colspan="2">Learners should investigate quadratic equations by writing what they know and factorizing, and investigating binomial expansion, compound proportion, mixtures and rate of work. They should learn that a vector has magnitude and direction and investigate vector algebra, mid-point of vector in algebraic expression, sequence and series.

Learners should explain arithmetic and geometric progression and derive the formulae and understand matrices in transformation on the Cartesian plane. They should identify inverse, determinant of matrices, shear and stretch, isometric and non-isometric transformations and their application.</td><td>• How can we use binomial expansion, compound proportion, mixtures and rate of work?
• How can we use algebra to express a midpoint of a vector?
• What are sequences and series?
• How can we derive the formulae for arithmetic and geometric progression?
• How do we determine identity, inverse and determinant of a matrix?
• How do we apply shear, stretch, isometric and non-isometric transformations?</td></tr>
<tr><td colspan="3">Learning outcomes</td></tr>
<tr><td>Knowledge and understanding</td><td>Skills</td><td>Attitudes</td></tr>
</table>

Understand quadratic equations (II): - Binomial expansion - Compound proportion - Mixtures and rate of work Understand vectors (II): - Vector algebra, mid-point of vector in algebraic expression - Sequences and series - Explain arithmetic and geometric progression - Derivation of the formulae for A.P. and G.P. Understand matrices (II): - Transformation on the Cartesian plane Understand identity and inverse; - Determinant of matrices, shear and stretch, Isometric and non-isometric transformation and their application	- Use quadratic methods to solve quadratic equations - Perform and carry out derivation of A.P. and G.P formulae - Identify different types of transformation on the Cartesian plane	- Appreciate and value quadratic methods in solving problems - Emphasize matrices as a useful tool in transformation

Contribution to the competencies:

Creative thinking through formulation of formulae and identification of types of transformation

Co-operation and communication in teams

Links to other subjects:

Physics: Vectors

Business: Matrices in calculations

Quadratic equations: Factorization

Teacher should check for understanding of the "sum" and "product" method. They should show steps before arriving at the answer. Note that the order of the brackets does not matter.

Exercise 1

1.

a) $(x+1)(x+2)$

b) $(x-2)(x-4)$

c) $(x+1)(x-9)$

d) $(x+a)(x+b)$

e) $(2x+3)^2$

f) $(3x+2)^2$

g) $(5x-1)(x+3)$

h) $(x-7)^2$

2.

a) $(x-5)^2 = 0,\ x = 5$

b) $(x+2)(2x-3) = 0, x = -2\ or\ x = \frac{3}{2}$

c) $(x-4)(x+4) = 0, x = -4\ or\ x = 4$ (Product and sum method does not work here. Students should remember "difference of 2 squares"

d) $(x-7)^2 = 0, x = 7$

Perfect squares

Exercise 2

1 b) i) $(\frac{12}{2})^2 = 36$

ii) $(\frac{16}{2})^2 = 64$

iii) $(\frac{-8}{2})^2 = 16$

iv) $(\frac{b}{2})^2 = \frac{b^2}{4}$

c i)20 ii) 8 iii) $2\sqrt{c}$

Solving a quadratic equation by completing the square:

Exercise 3

Students should carefully the steps in the example to solve this. The teacher should check for understanding. Students should remember that the square root of a number is both positive and negative. Answers can be left as surds or square roots can be read from the tables or calculators can be used.

1 a) $x = -1$

b) $x = -4 \; or \; x = 9$

c) $x = 4 + \sqrt{3} \; or \; 4 - \sqrt{3}$ (5.73 or 2.27)

d) $x = 4 + \sqrt{46} \; or \; 4 - \sqrt{46}$ (10.8 or -2.78)

2. Students should remember to divide by the coefficient of a first

a) $x = -1$

b) $x = 1.27 \; or \; x = -2.77$

c) $3x^2 + 7x - 4 = 0$ divide everything by 3

$$x^2 + \frac{7}{3}x - \frac{4}{3} = 0$$

$$x^2 + \frac{7}{3}x = \frac{4}{3}$$

Find $(\frac{b}{2})^2 = (\frac{7}{6})^2 = \frac{49}{36}$. Add this on both sides

$$x^2 + \frac{7}{3}x + \frac{49}{36} = \frac{4}{3} + \frac{49}{36}$$

$\left(x+\frac{7}{2}\right)^2=\frac{97}{36}$, get the square root on both sides and solve of x.

$x=0.475\ or\ x=-2.81$

d) $x=3\ or\ x=\frac{3}{2}$

e) Students should be able to derive the quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Solving quadratic equations using the Quadratic Formula

The last question $ax^2+bx+c=0$ gives us the **quadratic formula**

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-(4ac)}}{2a}}$$

Let's complete the square on the general equation and see exactly how that produces the Quadratic Formula. Recall the process of completing the square.

- Start with an equation of the form $x^2+bx+c=0$.
- Rewrite the equation so that x^2+bx is isolated on one side.
- Complete the square by adding $\left(\frac{b}{2}\right)^2$ to both sides.
- Rewrite the perfect square trinomial as a square of a binomial.
- Use the square root property and solve for x.

Can you complete the square on the general quadratic equation $ax^2+bx+c=0$? Try it yourself before you continue to the example below.

Hint: Notice that in the general equation, the coefficient of x^2 is not equal to 1. You can divide the equation by a, which makes some of the expressions a bit messy, but if you are careful, everything will work out, and at the end, you'll have the Quadratic Formula!

Example

Complete the square of $ax^2 + bx + c = 0$ to arrive at the Quadratic Formula.

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	Divide both sides of the equation by a, so that the coefficient of x^2 is 1.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Rewrite so the left side is in form $x^2 + bx$ (although in this case bx is actually $\frac{b}{a}x$).
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	Since the coefficient on x is $\frac{b}{a}$, the value to add to both sides is $\left(\frac{b}{2a}\right)^2$.
$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	Write the left side as a binomial squared.
$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$	Evaluate $\left(\frac{b}{2a}\right)^2$ as $\frac{b^2}{4a^2}$.
$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$	Write the fractions on the right side using a common denominator.

$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$	Add the fractions on the right.
$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$	Use the Square Root Property. Remember that you want both the positive and negative square roots!
$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$	Subtract $\frac{b}{2a}$ from both sides to isolate x.
$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$	The denominator under the radical is a perfect square, so: $\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}=\frac{\sqrt{b^2-4ac}}{2a}$
Answer $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Add the fractions since they have a common denominator.

There you have it, the Quadratic Formula.

Using the quadratic formula:

Consider $2x^2+4x+2=0,\ a=2, b=4\ and\ c=2$

$$\boldsymbol{x=\frac{-4\pm\sqrt{4^2-(4\times2\times2)}}{2\times2}}$$

$$x=\frac{-4\pm0}{4}$$

$$\boldsymbol{x=-1}$$

Exercise 4

1. a) $x = 0.46 \; or \; x = -6.54$

b) $x = 1.46 \; or \; x = 0.46$

c) $x = 1 \; or \; x = 3$

d) $x = -4.76 \; or \; x = -0.734$

2. a) $(x + 2)^2 + (4x + 4)^2 = 25x^2$

$2x^2 - 9x - 5 = 0$

$(x - 5)(2x + 1) = 0$

$x = 5 \; or \; x = -0.5 \quad \boldsymbol{x = 5m}$

b) $\boldsymbol{2x^2 - 2x - 8 = 0}$

$x = 1.56 \; or - 1.56$

Length = 3 x 1.56=4.68m

Area=7.3m^2

Application of quadratics to real life situations

Let learners talk about this.

Quadratic equations lend themselves to modeling situations that happen in real life, such as the rise and fall of profits from selling goods, the decrease and increase in the amount of time it takes to run a mile based on your age, and so on.

The wonderful part of having something that can be modeled by a quadratic is that you can easily solve the equation when set equal to zero and predict the patterns in the function values.

The vertex and x-intercepts are especially useful. These intercepts tell you where numbers change from positive to negative or negative to positive, so you know, for instance, where the ground is located in a physics problem or when you'd start making a profit or losing money in a business venture.

The vertex tells you where you can find the absolute maximum or minimum cost, profit, speed, height, time, or whatever you're modeling.

Exercise 5

1. The building is 64 feet tall, the ball peaks at 100 feet, and it takes 4 seconds to hit the ground.
   The ball is thrown from the top of the building, so you want the height of the ball when $t = 0$. This number is the initial t value (the y-intercept). When $t = 0$, $h = 64$, so the building is 64 feet high.
   The ball is at its highest at the vertex of the parabola. Calculating the $t$value, you get that the vertex occurs where $t = 1.5$ seconds. Substituting $t = 1.5$ into the formula, you get that $h = 100$ feet.
   The ball hits the ground when $h = 0$. Solving $-16t^2 + 48t + 64 = 0$, you factor to get $-16(t - 4)(t + 1) = 0$. The solution $t = 4$ tells you when the ball hits the ground.
   The $t = -1$ represents going backward in time, or in this case, where the ball would have started if it had been launched from the ground — not the top of a building.

2. Lado loses SSP100 (earns –SSP100) if he sells 0, needs to sell 13 to break even, and can maximize profits if he sells 1,006 umbrellas.
   If Lado sells no umbrellas, then $x = 0$, and he makes a negative profit (loss) of SSP 100. The break-even point comes when the profit changes

from negative to positive, at an x-intercept. Using the quadratic formula, you get two intercepts: at $x = 2,000$ and x is approximately 12.35.

The first (smaller) x-intercept is where the function changes from negative to positive. The second is where the profit becomes a loss again (too many umbrellas, too much overtime?). So, 13 umbrellas would yield a positive profit — he'd break even (have zero profit).

The maximum profit occurs at the vertex. Using the formula for the x-value of the vertex, you get that x is approximately 1,006.17. Substituting 1,006 into the formula, you get 4,000.1542; then substituting 1,007 into the formula, you get 4,000.15155.

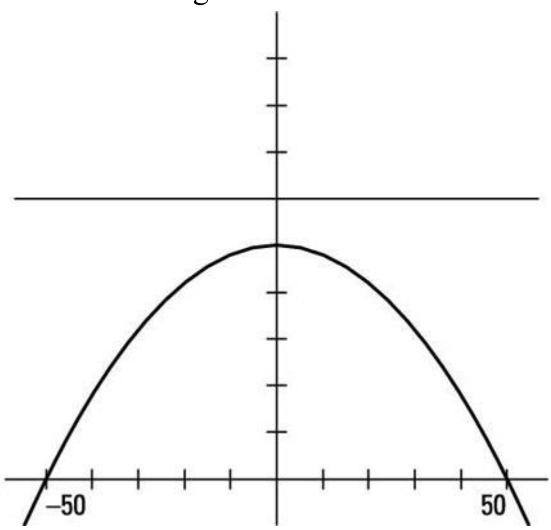
You see that Lado gets slightly more profit with 1,006 umbrellas, but that fraction of a cent doesn't mean much. He'd still make about SSP 4,000.

3. Wani took 40 seconds the first time; his best time was 8 seconds. Because the variable a represents the number of the attempt, find $T(1)$ for the time of the first attempt. $T(1) = 40$ seconds. The best (minimum) time is at the vertex. Solving for the a value (which is the number of the attempt),

$$a = \frac{-(-9)}{2(0.5)} = 9$$

He had the best time on the ninth attempt, and $T(9) = 8$.

4. The underpass is 50 feet high and 100 feet wide.



The highest point occurs at the vertex:

$$x = \frac{-0}{2(-0.02)} = 0$$

The x-coordinate of the vertex is 0, so the vertex is also the y-intercept, at (0, 50). The two x-intercepts represent the endpoints of the width of the overpass. Setting $50 - 0.02x^2$ equal to 0, you solve for x and get $x = 50, -50$. These two points are 100 units apart — the width of the underpass.

Binomial expansion

Group activity

Expression	Expansion	Coefficients
$(a+b)^0$	1	1
$(a+b)^1$	$a+b$	1 1
$(a+b)^2$	$a^2+2ab+b^2$	1 2 1
$(a+b)^3$	$a^3+3a^2b+3ab^2+b^3$	1 3 3 1
$(a+b)^4$	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$	1 4 6 4 1
$(a+b)^5$	$a^5+5a^4b+10a^3b^2+10a^2b^3 + 5ab^4+b^5$	
$(a+b)^6$	$a^6+6a^5b+15a^4b^2+20a^3b^3 + 15a^2b^4+6ab^5+b^6$	

Students working in groups should be encouraged to state any observation they make after expanding the various binomial expansions. These observations include:

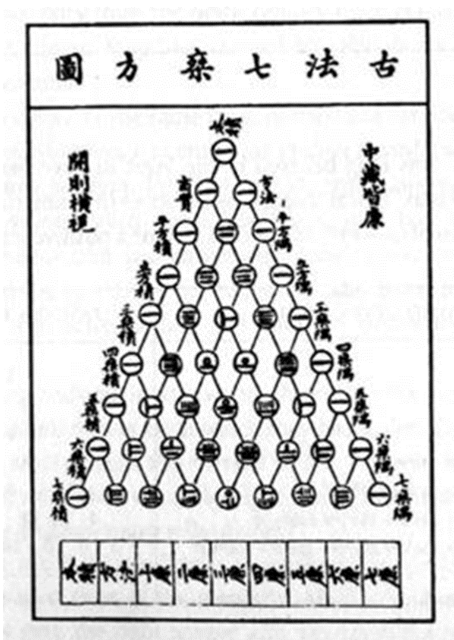
1. Each row starts with 1
2. Each of the numbers in the row is obtained by adding the two numbers above it.
3. The power of a reduces as the power of b increases

4. The number of terms is one more than the power being expanded e.g. in the expansion of
   $(a+b)^3$, the number of terms is 4
5. The sum of the powers for a and b in each term is equal to the power being expanded. e.g.
   $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ each of term of this expansion has a sum of powers equal to 4
6. As students expand $(a+b)^7$, guide the students to integrate the observations noted above. Start with the coefficients from Pascal's triangle, then make sure there are 8 terms and the powers are all adding up to 7. $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

The Addition Triangle

Task

Here is an ancient Chinese copy of the triangle. In groups, decipher the symbols used by the ancient Chinese for numbers.



Exercise 5

1. a) $a^3 - 3a^2b + 3ab^2 - b^3$

   b)$8x^3 + 12x^2y + 6xy^2 + y^3$

   c)$1 - 8b + 24b^2 - 32b^3 + 16b^4$

   d)$16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 27y^3$

e)$x^6 - 18x^5y + 135x^4y^2 - 540x^3y^3 + 1215x^2y^4 - 1458xy^5 + 729y^6$

2. a)$1 + 3\sqrt{5} + 15 + 5\sqrt{5}$

   b) $56 - 24\sqrt{5}$

   c)$217 + 104\sqrt{3}$

   d) $44762 - 19707\sqrt{7}$

3. a) $x^3 + \frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{8}$

   b) $\frac{81}{16}y^4$

   c) $16x^4 - 96x^3 + 216x^2 + 24x + 81$

   d) $27x^3 - 9x^2 + xy^2 + \frac{1}{2}y^3$

Investigation

Guide students in conducting the following investigation.

Part A: Binomial Expansions and Pascal's Triangle.

1. . Expand and simplify each of the following. When finished check with your partner that you have expanded correctly.

(1) $(a+b)^1$ (2) $(a+b)^2$ (3) $(a+b)^3$ (4) $(a+b)^4$.

2. In the triangle of numbers below:

				1					R_0
			1		1				R_1
		1		2		1			R_2
	1		3		3		1		R_3
1		4		6		4		1	R_4

(1) Look for the patterns from row to row and explain how to write down the next row of the triangle.
(2) Make your own copy of the triangle with at least seven rows.
(3) Write down the third row of the triangle. Find and write down the connection between the expansion of $(a+b)^3$ and the third row of the triangle.
(4) Use the results of (3) to expand:
  (a) $(a+b)^5$
  (b) $(a+b)^6$
  (c) $(a+b)^7$

3. Use the Addition triangle to expand any 3 of the following. The first row of this question allows you to demonstrate satisfactory understanding while the second row questions are more challenging and allow you to demonstrate deeper understanding.

(1) $(x+2)^3$ (2) $(x+3)^3$ (3) $(x+5)^3$ (4) $(x-1)^4$

(5) $(2x+3)^3$ (6) $(2x+1)^4$ (7) $(3x-4)^3$ (8) $(2-3x)^4$.

Part B: The Addition Triangle and probability

4. (1) Write down the possibilities when two coins are tossed. How many possibilities are there?

(2) Find the total of row 2 of the triangle.

(3) Find the probability of tossing two coins and obtaining:

(a) No tails

(b) One tail

(c) Two tails

(4) How do these probabilities relate to the triangle? Write a sentence on this.

(5) Write down the possibilities when three coins are tossed. How many possibilities are there?

(6) Find the probability of tossing three coins and obtaining:

(a) No heads.

(b) One head.

(c) Two heads.

(d) Three heads.

(7) Use the triangle to find the following probabilities.

(a) Tossing 4 coins and obtaining 3 tails.

(b) Tossing 6 coins and obtaining 5 heads.

(c) Tossing 6 coins and obtaining 4 tails.

(8) Make up and answer three probability questions of your own that relate to the addition triangle.

Exercise: Application of binomial expansion

1. a) 1.0824

   b) 1.0120

   c) 0.857

   d) 15.6824

2. a) $1 - 8x + 24x^2$

   b) 0.9224

3. a) $64 + 576x + 2160x^2 + 4320x^3 + 4860x^4$

   b) 83.3446

Compound proportions, mixtures and rates of work:

Exercise 6

1. Working 8 hours per day, a glass factory makes 6000 bottles in 3 days. How long would it take to make 10000 bottles working 9 hours per day?
2. I paid SSP 30,000 for 12 chairs. If my friend Charles want 5 chairs as mine. How much does he have to pay?
3. If 5 trucks transport 120 tons of goods in 2 days: what goods quantity will 7 trucks transport in 3 days?
4. It takes 15 days for a team of 10 workers working 8 hours a day to finish an order. How many people with part time jobs (half the day) will be needed to realize the same work in 10 days?
5. Three gardeners mow the grass of a park in 12 hours. How long will it take if one of them have to go?

Solutions:

1) 4 days 10 hours
2) SSP 1250
3) 6 hours

4) 252 tons
5) 30 people

Exercise 7: Discuss the following questions in groups and show the correct working

1. Tractor A takes 4 days to mow a field while tractor B takes 7 days to mow the same field. How long will it take both tractors to do the work?
2. Two men each working for 8 hours a day can cultivate an acre of land in 4 days. How long would 6 men, each working 4 hours a day, take to cultivate 4 acres?
3. A shopkeeper buys 1 kg of maize flour costs SSP 100 and 1 kg of millet flour costs SSP 150. In what proportion should they be mixed so that 1 kg of the mixture costs SSP 140? How much should the mixture be sold so that the shopkeeper makes a 10% profit?
4. A cold water tap fills a tub in 10 min and a hot water tap fills the tub in 15 min. How long would they take to fill the tub if both are turned on simultaneously?

Answers

1. $2\frac{6}{11} days$
2. $10\frac{2}{3} days$
3. a) 1:4
   b) SSP154
   c) 6 min

Vectors II

Exercise 8

The following are answers to the practice questions:

1. Magnitude 8.6, angle 54 degrees
   Apply the equation

$$v = \sqrt{x^2 + y^2}$$

to find the magnitude, which is 8.6.
Apply the equation theta = $\tan^{-1}(y/x)$ to find the angle: $\tan^{-1}(7.0/5.0)$ = 54 degrees.

2. Magnitude 18.4, angle 45 degrees
   Apply the equation
   $$v = \sqrt{x^2 + y^2}$$
   to find the magnitude, which is 18.4.
   Apply the equation theta = $\tan^{-1}(y/x)$ to find the angle: $\tan^{-1}(13.0/13.0)$ = 45 degrees.

3. Magnitude 1.4, angle 135 degrees
   Apply the equation
   $$v = \sqrt{x^2 + y^2}$$
   to find the magnitude, which is 1.4.
   Apply the equation theta = $\tan^{-1}(y/x)$ to find the angle: $\tan^{-1}(1.0/-1.0)$ = –45 degrees.
   However, note that the angle must really be between 90 degrees and 180 degrees because the first vector component is negative and the second is positive. That means you should add 180 degrees to –45 degrees, giving you 135 degrees (the tangent of 135 degrees is also 1.0/–1.0 = –1.0).

4. Magnitude 8.6, angle 234 degrees
   Apply the equation
   $$v = \sqrt{x^2 + y^2}$$
   to find the magnitude, which is 8.6.
   Apply the equation $theta = tan^{-1}(\frac{y}{x})$ to find the angle: $tan^{-1}(\frac{-7.0}{-5.0}) = 54\ degrees$.
   However, note that the angle must really be between 180 degrees and 270 degrees because both vector components are negative. That means you

should add 180 degrees to 54 degrees, giving you 234 degrees (the tangent of 234 degrees is also $\frac{-7.0}{-5.0} = \frac{7.0}{5.0}$.

Sequences and series

Exercise 9

Ensure that students understand how to find the n^{th} term. Use the examples to check for students' understanding on finding the nth term.

a) 1, 2, 3, 4, 5, 6, 7,……(adding 1 to the previous number) **n, 50**

b) 3, 7, 11, 15, 19, 23, 27,31,…..(adding 4 to the previous number) **4n-1, 199**

c) 0, 3, 8, 15, 24, 35, 48,……(adding consecutive odd numbers) $\boldsymbol{n^2-1}$**, 2499**

Series

Exercise 10

Write downS_1,S_2,...,S_n for the sequences

1. 1,3,5,7,9,11;
2. 4, 2, 0, −2, -4.

Arithmetic Progression

Investigation

1a) $11 = 8 + 3$

b) $14 = 8 + (3 + 3)$

c) $17 = 8 + (3 + 3 + 3)$

Notice that the second term is equal to the first term plus the difference, the third term is equal to the first term plus twice the common difference, the fourth term is equal to the first term plus thrice the difference.

d)

$2nd\ term = 8 + (2-1)3$, $3rd\ term = 8 + (3-1)3$, therefore $nth\ term = a + (n-1)d$

$4th\ term = 8 + (4-1)3 = 8 + (3 \times 3) = 17$

e) 3, 5, 7, 9,……..

$a = 3, d = 2$ therefore

$$10th\ term = 3 + (10-1)2 = 10 + (9 \times 2) = \mathbf{28}$$

Exercise 11

1. Write down the first five terms of the AP with first term 8 and common difference 7.
2. Write down the first five terms of the AP with first term 2 and common difference −5.
3. What is the common difference of the AP 11, −1, −13, −25...?
4. Find the 17th term of the arithmetic progression with first term 5 and common difference 2.
5. Write down the 10th and 19th terms of the AP:
   i) 8, 11, 14, …
   ii) 8,5,2....
6. An AP is given by k, $2\frac{k}{3}$, $\frac{k}{3}$, 0,....
   (i) Find the sixth term.
   (ii) Find the n th term.
   (iii) If the 20th term is equal to 15, find k.
7. For each sequence:
   i) Find the 15th term.
   ii) Find an expression for the nth term.
      a) 3, 6, 9…
      b) 25, 40, 55…
      c) 36, 41, 46 …
      d) 100, 87, 74 …
8. Find the number of terms in each sequence.

a) 5, 10, 15… 255

b) 4.8, 5.0, 5.2… 38.4

c) $\frac{1}{2}, \frac{7}{8}, \frac{5}{4}, \dots \dots \dots 14$

d) 250, 221, 192… -156

9. An arithmetic progression has first term 19 and 15th term 31.6. Find the common difference.
10. The second term of an arithmetic sequence is 7 and the ninth term is 28. Find the common difference and the first term of the sequence.

Arithmetic series

Students should be made aware that a series is the sum of the terms of a sequence.

Investigation

Arrange the students in pairs and let them follow the steps in the investigation to get the sum of n terms an arithmetic series. Check for correct working.

Discussion questions

Students should be in groups of at least 4. Allow discussion and working can be written on exercise books or manila paper. All members of the group should take part in the presentation.

Exercise 12

1. Find the sum of the first 23 terms of the AP $4, -3, -10 \dots$
2. An arithmetic series has first term 4 and common difference $\frac{1}{2}$. Find
   (i) the sum of the first 20 terms,
   (ii) the sum of the first 100 terms.
3. Find the sum of the arithmetic series with first term 1, common difference 3, and last term 100.

4. The sum of the first 20 terms of an arithmetic series is identical to the sum of the first 22 terms. If the common difference is -2, find the first term.

Investigation: saving money

Joel decides to start saving money. He saves SSP 20 the first week, SSP 25 the second week, SSP 30 the third week, and so on.

Week number	Weekly savings	Total savings
1	20	20
2	25	45
3	30	75
4	35	110
5	40	150
9	45	195
7	50	245
8	55	300

b) Let students find ways of getting this answer

10^{th} week: SSP 65, 17^{th} week: SSP 105

c) A year has 52 weeks; students should use the 3^{rd} column in the table to find the total saved in 52 weeks. SSP 7670

d) Use the table to find when he saves a total of SSP 1000

e) $m = 5n + 15$

f) 20,45,75,110,…1^{st} difference = 25,30,35,….2^{nd} difference = 5 nth term = $2.5n^2 + 17.5n$

Geometric Progression

Investigation: To be done in groups

1. Teacher should ensure students are able to identify the first term and the common ratio.

$$a = 1, r = 5$$

2. The discussion among the students should finally get to:

$2nd\ term = 1 \times 5$

3. $3rd\ term = 1 \times 5 \times 5$
4. $4th\ term = 1 \times 5 \times 5 \times 5$

Teacher should let students see the pattern:

$$5th\ term = 1 \times 5 \times 5 \times 5 \times 5\ nth\ term = a \times r^{n-1}$$

5. $10th\ term = 1 \times 5^{10-1} = 5^9 = 1\,953\,125$
6. Students should present this answer to the whole group. Teacher to encourage creativity and ensure the sequences are correct.

Exercise 13

1. Write down the first five terms of the geometric progression which has first term 1 and common ratio $\frac{1}{2}$.
2. Find the 10th and 20th terms of the GP with first term 3 and common ratio 2.
3. Find the 7th term of the GP 2, −6, 18,...,

Investigation

Students should be in groups of at least 4. Allow discussion and working can be written on exercise books or manila paper. All members of the group should take part in the presentation.

Exercise 14

1. Step 1: Identify the first term and the common difference (a and d)

Step 2: Use the formula for finding the nth term: $nth\ term = a + (n-1)d$

a) 3, 6, 9….

$$15^{th}\ term = 3 + (15-1)3 = 3 + 42 = \mathbf{45}$$

$$n^{th}\ term = 3 + (n-1)3 = 3 + 3n - 3 = \mathbf{3n}$$

b) 25, 40, 55,…..

$$15^{th}\ term = 25 + (15-1)15 = 25 + 210 = \mathbf{235}$$

$$n^{th}\ term = 25 + (n-1)15 = 25 + 15n - 15 = \mathbf{10 + 15n}$$

c) 36, 41, 46, … **(Note that the difference is negative)**

$$15\ ^{th}term = 36 + (15-1)(-5) = 36 - 70 = \mathbf{-34}$$

$$n^{th}\ term = 36 + (n-1)(-5) = 36 - 5n + 5 = \mathbf{41 - 5n}$$

d) 100,87,74,……..

$$15^{th}\ term = 100 + (14-1)(-13) = 100 - 169 = \mathbf{-69}$$

$$n^{th}\ term = 100 + (n-1)(-13) = 100 - 13n + 13 = \mathbf{113 - 13n}$$

2. Use the example 2 to guide the students to find the answer

a) $n^{th}\ term = 5 + (n-1)5 = 5 + 5n - 5 = 5n$

$5n = 255, n = 51$, therefore there are **51 terms**

b)) $nth\ term = 4.8 + (n-1)(0.2) = 4.8 + 0.2n - 0.2 = 4.6 + 0.2n$

$$4.6 + 0.2n = 38.4$$

$$0.2n = 33.8$$

$n = 169$, therefore there are **169 terms**

c)) $n^{th}\ term = \frac{1}{2} + (n-1)\left(\frac{3}{8}\right) = \frac{1}{2} + \frac{3}{8}n - \frac{3}{8} = \frac{1}{8} + \frac{3}{8}n$

$$\frac{1}{8} + \frac{3}{8}n = 14$$

$$\frac{3}{8}n = \frac{111}{8}$$

$n = 37$, therefore there are **37 terms**

d) $n^{th}\ term = 250 + (n-1)(-29) = 250 - 29n + 29 = 279 - 29n$

$$279 - 29n = -156$$

$$-29n = -435$$

$n = 15$, therefore there are **15 terms**

3. $a = 19, 15th\ term = 31.6$

$15^{th}\ term = 19 + (15-1)(d) = 31.6$

$19 + 14d = 31.6$

$14d = 12.6, \boldsymbol{d = 0.9}$

4. $2nd\ term = 7, 9th\ term = 28$

$2nd\ term = a + (2-1)(d) = 7$ and $9th\ term = a + (9-1)(d) = 28$

$a + d = 7$ and $a + 8d = 28$ (solve these equations simultaneously)

$\boldsymbol{d = 3, a = 4}$

5. $3rd\ term = a + (3-1)d = a + 2d = a + 20$

$2nd\ term = a + (2-1)d = a + 10$

$(a+20)(a+10) = 0$

$a + 20 = 0\ or\ a + 10 = 0$

$\boldsymbol{a = -20\ or\ a = -10}$

Geometric Series

In groups, students should follow the steps in the investigation and come up with the formula for the sum of n terms of a geometric series

Matrices & transformations

Review previous work on matrices: addition, subtraction, multiplication by a scalar and finding the determinant and inverse of a matrix. Use examples to test for understanding.

Review previous work on transformations of objects: reflection, rotation, enlargement and translation. Use examples to test for understanding.

Review work on vectors: position and column vectors

Investigation

Divide the class into groups.

Let students draw shapes on graph or squared paper with an appropriate scale for the Cartesian plane.

Students should **pre multiply** the position vector by the matrix, teacher should check whether correct matrix multiplication is being done.

Encourage discussion on the different types of transformations and how to describe each fully.

1 a) Reflection in the line$x = 0$ (y-axis)

2. b) Rotation of 90^0 anticlockwise about (0,0)

3. c) Reflection in the line $y = x$

4. d) there is no change – teacher should check that students can identify that the matrix given is the **identity matrix.** Review the characteristics of the identity matrix.

Finding the matrix of transformation

Activity: Complete in pairs.

Arrange students in pairs check that they follow the steps given. Review solving of simultaneous equations.

1. $a = 1, b = 0, c = 0, d = -1$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Isometric and non-isometric transformations

Teacher to ensure students understand the difference between isometric and non-isometric transformations. Review characteristics of each type of transformation again: rotation, reflection, translation and enlargement.

Shear: Task

1. Divide students in pairs and have them draw the triangles. What can they notice?

Points A and B have not changed, point C has moved. The image looks different from the object.

x-axis is invariant – any point on this line does not move

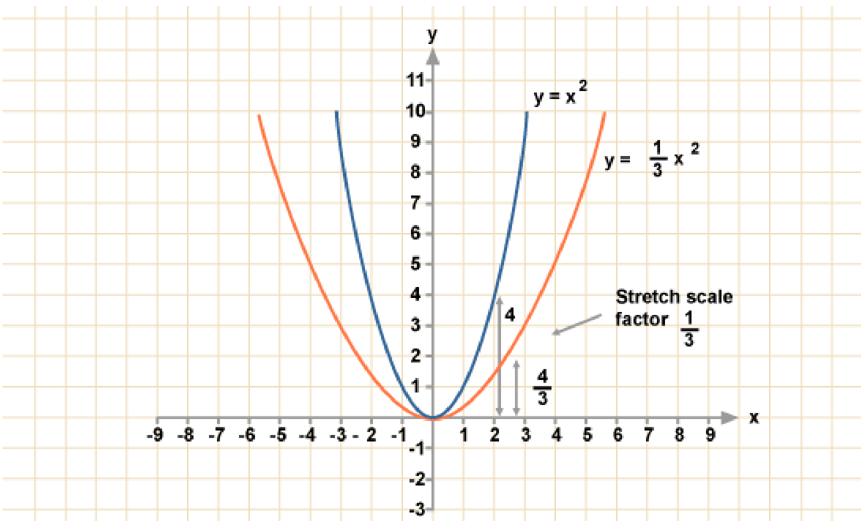
Students should understand how to identify the invariant line and to state the shear factor.

2. x-axis is invariant, shear factor is $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

3. y-axis invariant, shear factor is $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Stretch: Task

A stretch, also known as a one-way enlargement, is defined parallel to a specified or direction. Any line parallel to this direction is invariant, and there will be one line of invariant points perpendicular to this direction. Points of the plane are moved so that their distances from the line of invariant points are increased by a factor of k. Distances are, in general, not preserved and areas are increased by a factor of k.



1. Invariant line is the x- axis($y = 0$), stretch factor is 2 (y co-ordinates of points P and Q are multiplied by 2) because there is no point on the invariant line, all x co-ordinates are multiplied by 2. y-co-ordinates do not change. P'(4,3), Q'(8,3), R'(8,0) and S'(6,0).

3. Let student come up with their own shapes and show understanding of shear and stretch. Let them present their findings for number 3 to the whole class.

UNIT 4: CALCULUS

Math Secondary 3			**Unit 4: Calculus**
Learn about			**Key inquiry questions**
Learners should investigate how to differentiate a function and know that differentiation is the opposite operation of integration. They should explain the notation used in differentiation dy/dx and f'(x) and learn about types of functions, e.g. quadratic, cubic, reciprocal, exponential and trigonometric functions. Learners should find the gradient of a curve at a point as well as the gradient of $y = x^n$, investigate how a gradient is related to tangent, and explain how to find the gradient of a curve at a point.			- How do we explain differentiation? - How do you find the gradient of a curve at a point? - How do you find the gradient of $y = x^n$
Learning outcomes			
Knowledge and understanding	**Skills**		**Attitudes**
- Understand differentiation: gradient of the curve at a point, gradient of $y = x^n$	- Investigate the concept of differentiation and how it relates to the tangent of the angle of the curve at that point - Find the gradient of a curve at a point as well as the gradient of $y = x^n$ - Investigate how a gradient is related to tangent,		- Appreciate using differentiation when solving calculus problems

Contribution to the competencies: Critical thinking through analysis and investigation Co-operation and communication through teamwork
Links to other subjects: Physics Chemistry Business

Differentiation

How to Differentiate a Function

A function expresses relationships between constants and one or more variables. For example, the function $f(x) = 5x + 10$ expresses a relationship between the variable x and the constants 5 and 10. Known as derivatives and expressed as dy/dx, $df(x)/dx$ or $f'(x)$, differentiation finds the rate of change of one variable with respect to another -- in the example, $f(x)$ with respect to x.
Differentiation is useful for finding the optimal solution, meaning finding the maximum or minimum conditions.
Some basic rules exist with regard to differentiating functions.

Differentiate a constant function.

The derivative of a constant is zero. For example, if $f(x) = 5$, then $f'(x) = 0$.

Apply the power rule to differentiate a function. The power rule states that if $f(x) = x^n$ or x raised to the power n, then $f'(x) = nx^{(n-1)}$ or x raised to the power $(n - 1)$ and multiplied by n. For example, if $f(x) = 5x$, then $f'(x) = 5x^{(1-1)} = 5$. Similarly, if $f(x) = x^{10}$, then $f'(x) = 9x^9$; and if $f(x) = 2x^5 + x^3 + 10$, then $f'(x) = 10x^4 + 3x^2$.

Find the derivative of a function using the product rule.

The differential of a product is not the product of the differentials of its individual components: If $f(x) = uv$, where u and v are two separate functions, then $f'(x)$ is not equal to $f'(u)$ multiplied by $f'(v)$. Rather, the derivative of a product of two functions is the first multiplied by the derivative of the second, plus the second multiplied by the derivative of the first. For example, if $f(x) = (x^2 + 5x)(x^3)$, the derivatives of the two functions are $2x + 5$ and $3x^2$, respectively. Then, using the product rule,

$f'(x) = (x^2 + 5x)(3x^2) + (x^3)(2x + 5)$

$= 3x^4 + 15x^3 + 2x^4 + 5x^3$

$= 5x^4 + 20x^3.$

Get the derivative of a function using the quotient rule

A quotient is one function divided by another. The derivative of a quotient equals the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, then divided by the denominator squared. For example, if $f(x) = (x^2 + 4x) / (x^3)$, the derivatives of the numerator and the denominator functions are $2x + 4$ and $3x^2$, respectively. Then, using the quotient rule,

$$f'(x) = \frac{[(x^3)(2x + 4) - (x^2 + 4x)(3x^2)]}{(x^3)^2}$$

$$= \frac{(2x^4 + 4x^3 - 3x^4 - 12x^3)}{x^6}$$

$$= \frac{(-x^4 - 8x^3)}{x^6}.$$

Use common derivatives. The derivatives of common trigonometric functions, which are functions of angles, need not be derived from first principles -- the derivatives o

f sin x and cos x are cos x and -sin x, respectively. The derivative of the exponential function is the function itself -- $f(x) = f'(x) = e^x$, and the derivative of the natural logarithmic function, ln x, is $\frac{1}{x}$. For example, if $f(x) = sin\, x + x^2 - 4x + 5$, then $f'(x) = cos\, x + 2x - 4$.

Investigation

1. Students should be made aware of the difference between secant and tangent lines. Remind them on the gradient formula.

2. As students fill in coordinates for Point Q, they should become aware that the value of the y coordinate is the square of the x coordinate.

Gradient for the last point in the table should be:

$$\frac{(x+h)^2 - x^2}{(x+h) - x} = \frac{x^2 + 2xh + h^2 - x^2}{x + h - x}$$

$$= \frac{2xh + h^2}{h} = \mathbf{2x + h}$$

c) When h is too small, the gradient is $\mathbf{2x}$

3. Students should complete a table of at least 4 values for $y = x^3$ before getting the last value.

The gradient Of PQ is

$$\frac{(x+h)^3 - x^3}{(x+h) - x} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{x + h - x}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h} = \mathbf{3x^2 + 3xh + h^2}$$

When h becomes too small, gradient is $\boldsymbol{3x^2}$

4. Learners should use the same procedure as question 2 and 3. Guide them to see the pattern emerging from the function and the gradient.

$$\frac{[3(x+h)+4]-(3x+4)}{(x+h)-x} = \frac{3x+3h+4-3x-4}{x+h-x}$$

$$= \frac{3h}{h} = \mathbf{3}$$

Gradient is **3**

5. Students should see the connection between the rule and what they have been getting in the investigation.

i) $y = x^6$ is $\boldsymbol{5x^5}$ ii) $y = 3x^2$ is $6x$ iii)$y = x^{-2}$ is $-2x^{-1}$

Check for understanding of the rule.

Rules of differentiation

Types of functions

a) Quadratic functions

A **quadratic expression** is an expression of the form:

$ax^2 + bx + c$,

where x is a variable and a, b, and c are constants, and a is not equal to zero.

The term a x^2 is called the **quadratic term**, bx is called the **linear term** and c is called the **constant term**.

The constant a is called the **leading coefficient**, b is called the **linear coefficient**, and c is called the **additive constant**.

Example: These are quadratic expressions:

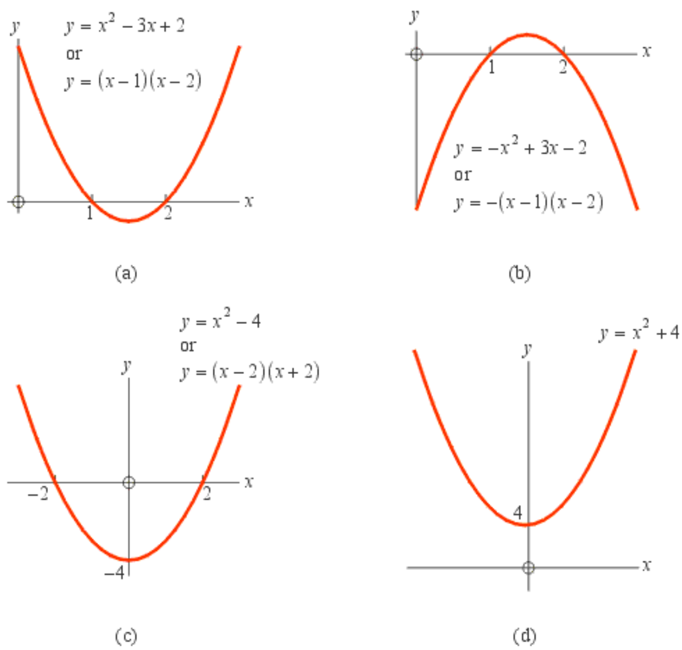
- $x^2 - 3x + 2$
- $-5x^2 + 7$

We can create a **quadratic function** called, say *f*, whose input is *x* and whose output, $f(x)$, is the quadratic expression evaluated at *x*:

$$x \rightarrow \boxed{f} \rightarrow ax^2 + bx + c$$

Example: $f(x) = x^2 - 3x + 2$ is a quadratic function. If we let the input be, say $x = 4$, then the output is $f(4) = 4^2 - 3 \cdot 4 + 2 = 6$.

We can give the output of the quadratic function the name "*y*" and make a graph of *y* versus *x*. We will get a curve like the four curves shown below. These curves are called **parabolas**. They feature a point called the **vertex** where the parabola reaches its maximum height or depth and turns around. The parabola **opens upward** if $a > 0$ and **opens downward** if $a < 0$ but its exact location depends on the values of all three constants *a*, *b* and *c*. In the picture below, parabolas (a), (c) and (d) open upward and have their vertex at the bottom. Parabola (b) opens downward and has its vertex at the top.



(a) (b) (c) (d)

If we set the quadratic expression equal to zero or if we set $y = 0$ or $f(x) = 0$ then we get the so-called **quadratic equation**:

$a\,x^2 + b\,x + c = 0.$

(Note that setting $y = 0$ in the graph means that we are looking at points where the parabola crosses the x axis, and setting $f(x) = 0$ in the quadratic function means that we are looking for values of x for which the output of the quadratic function is zero.)

There is a close connection between:

- The values of x that cause a quadratic expression to equal zero.
- The places where a quadratic function's graph (the parabola) crosses the x axis.
- The roots or solutions of a quadratic equation.
- The factors of a quadratic expression.

b) Exponential functions

Exponential functions are closely related to geometric sequences. A **geometric sequence** is a list of numbers in which each number is obtained by *multiplying* the previous number by a fixed factor m. An example is the sequence {1, 3, 9, 27, 81, …}. If we label the numbers in the sequence as $\{y_0, y_1, y_2, \ldots\}$ then their values are given by the formula
$y_n = y_0 \cdot m^n.$

A geometric sequence is completely described by giving its starting value y_0 and the multiplication factor m. For the above example $y_0 = 1$ and $m = 3$. Another example of a geometric sequence is the sequence {40, 20, 10, 5, 2.5, …}. For this sequence $y_0 = 40$ and $m = 0.5$.

An **exponential function** is obtained from a geometric sequence by replacing the counting integer n by the real variable x. The graph below shows the exponential functions corresponding to these two geometric sequences.

Thus we define an exponential function to be any function of the form

$y = y_0 \cdot m^x.$

It gets its name from the fact that the variable x is in the exponent. The "*starting value*" y_0 may be any real constant but the base m must be a positive real constant to avoid taking roots of negative numbers.

The exponential function $y = y_0 \cdot m^x$ has these two properties:

- When $x = 0$ then $y = y_0$.

- When x is increased by 1 then y is multiplied by a factor of m. This is true for any real value of x, not just integer values of x. To prove this suppose that y has some value y_a when x has some value x_a. That is,

  $$y_a = y_0 \cdot m^{x_a}$$

  Now increase x from x_a to $x_a + 1$. We get

  $$y = y_0 \cdot m^{x_a+1} = y_0 \cdot m^{x_a} \cdot m^1 = y_a \cdot m$$

  We see that y is now m times its previous value of y_a. If the multiplication factor $m > 1$ then we say that y **grows exponentially**, and if $m < 1$ then we say that y **decays exponentially**.

Exercise 1

1 .a) $f'^{(x)} = 12x^2 + 4x$

b) $f'^{(x)} = \frac{3}{2}x$

c) $f'^{(x)} = 2x + 2$

d) $f'^{(x)} = 8x + 2 + 3x^{-2}$

2. a) 4

b) $3x^2 + 4x$

c) 0

3. a)) $3x^2 + 2x$

b)) $x + \frac{3}{4}$

Equation of tangents and normal to a curve

Exercise 2

Students should follow the steps and use previous knowledge of finding the equation of a line.

1 a) $y = 2x$

b) substitute $x = 1$ into the derivative, gradient=2

c) $y = 2x$

d) gradient of the normal = $-\frac{1}{2}$

e) $y = -\frac{1}{2}x + \frac{5}{2}$

2. a) $y = 5x - 1$

b) $y = -\frac{1}{5}x + \frac{21}{5}$

3. (-8,-28), $y = -\frac{1}{8}x - 29$

The equation of a normal to a curve

After going over the explanation provided in the student's book, guide learners in completing the following activity in pairs.

Exercise 3

2. For each of the functions given below determine the equations of the tangent and normal at each of the points indicated.

a) $f(x) = x^2 + 3x + 1$ at $x = 0$ and 4.

b) $f(x) = 2x^3 - 5x + 4$ at $x = -1$ and 1.

c) $f(x) = \tan x$ at $x = \frac{\pi}{4}$.

d) $f(x) = 3 - x$ at $x = -2$, 0 and 1.

3. Find the equation of each normal of the function $f(x) = \frac{1}{3}x^3 + x^2 + x - \frac{1}{3}$ which is parallel to the line $y = -\frac{1}{4}x + \frac{1}{3}$.

4. Find the x co-ordinate of the point where the normal to $f(x) = x^2 - 3x + 1$ at $x = -1$ intersects the curve again.

Investigation: to be done in groups

Guide learners in carrying out the following investigation.

1. Consider the function $y = x^2 + 1$.
    a) Find its gradient function (its derivative).
    b) What is the gradient of the tangent line at $x = 1$? This is the gradient of the tangent at $x = 1$.
    c) The tangent is passing thorough the point (1, 2), find its equation in terms of $y = mx + c$.
    d) What is the gradient of the normal line at this point?
    e) The normal line is also passing through point (1, 2), find its equation in terms of $y = mx + c$.
2. Use the same steps to find the equation of the tangent line and normal lines for the function $y = x^3 + 2x + 1$ at the point (1, 4)
3. Determine the point on the curve $y = -\frac{1}{2}x^2 + 4$ at which the gradient is 8. Hence, find the equation of the normal to the curve at this point.

Stationary points

Investigation: Stationary points

Students should notice when the gradient is zero and the change in sign before this point and after.

Characteristics of **maximum points, minimum points and point of inflection** should be discussed

Exercise 2:

1. (0,5), point of inflection
2. (1,1), minimum point
3. $(\frac{7}{4}, \frac{169}{8})$, minimum point
4. (1,-4),minimum point, (-1,4) maximum point
5. (0,3) point of inflection, (-3,165) maximum,(3,159) minimum

Kinematics

Investigation:

a)

t	0	1	2	3
S(t)	4	8	12	16

b) Guide learners in drawing the graph of the function on a graph paper. Check for proper graphs.

c) velocity is constant 4m/s

d) 4

e) velocity = derivative of the function

2. a) $derivative = 6t^2 + 8t - 8$

$substitute\ t = 2, v = 32m/s$

$substitute\ t = 3, v = 70\ m/s$

3. a)

t	0	1	2	3
$v(t)$	2	10	18	26

b) Guide learners in drawing the graph of velocity against time on a graph paper.

Check for proper graphs.

c) Find the gradient=8

d) $a = \frac{dv}{dt} = 8$

$substitute\ t = 2, acceleration = 8m/s^2$

$substitute\ t = 3, acceleration = 8m/s^2$

4. a) $a = \frac{dv}{dt} = 10t - 1$

$at\ t = 1, acceleration = 9m/s^2$

$at\ t = 3, acceleration = 29m/s^2$

b) $10t - 1 = 0, t = 0.1sec$

5. a) $initial\ height\ is\ when\ t = 0, 4m$

b) $substitute\ t = 2, height = 20m$

c) $\frac{ds}{dt} = -32t + 40$

d)$v = \frac{ds}{dt}, initial\ velovity, t = 0$

$-32(0) + 40 = 40m/s$

e) $v = 0 = -32t + 40$

$t = 1.25sec$

f) $maximum\ height = value\ of\ s\ at\ the\ stationary\ point$

The stationary point is v=0, t=1.25s

$substitute\ t = 1.25\ in\ s(t) = -16t^2 + 40t + 4$

$$s = 29m$$

6. $r = 5cm$

Integration

Integration is the reverse of differentiation.

However:

If $y = 2x + 3$, $\frac{dy}{dx} = 2$
If $y = 2x + 5$, $\frac{dy}{dx} = 2$
If $y = 2x$, $\frac{dy}{dx} = 2$

So the integral of 2 can be $2x + 3$, $2x + 5$, $2x$, etc.
For this reason, when we integrate, we have to add a constant. So the integral of 2 is $2x + c$, where c is a constant.

An "S" shaped symbol is used to mean the integral of, and dx is written at the end of the terms to be integrated, meaning "with respect to x". This is the same "dx" that appears in $\frac{dy}{dx}$.

To integrate a term, increase its power by 1 and divide by this figure. In other words:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

Example:

$$\int x^5 \, dx = \frac{1}{6} x^6 + c$$

When you have to integrate a polynomial with more than 1 term, integrate each term. So:

$$\int x^3 + 4x^2 + 3 \, dx$$

$$= \frac{x^4}{4} + \frac{4x^3}{3} + 3x + c$$

3 is $3x^0$, so the integral is $3x^1$

Investigation

1. Students should notice that the derived functions are all the same.
$$\frac{dy}{dx} = 2x + 2$$
2. Again the derived functions are all equal. Students should try to work backwards to get the function from the derived function. $\frac{dy}{dx} = x + 2$

   They should realize that

$$\frac{x^{1+1}}{1+1} + 2x^{0+1}\frac{2x^{0+1}}{0+1} + c = \frac{x^2}{2} + 2x + c = f(x)$$

Exercise 4

1a) $\frac{1}{3}x^6 + c$

b) $\frac{3}{2}x^2 + 4x + c$

c) $x^3 + x^2 + x + c$

d) $2x^{\frac{1}{2}} + c$

e) $\frac{3}{4}x^{\frac{4}{3}} + c$

2. $\frac{dy}{dx} = 3x^2 + 2\ therefore\ y = x^3 + 2x + c$

Substitute (1,-1), $y = x^3 + 2x - 4$

3. $y = \frac{2}{3}x^6 + 4x^2 + c$, substitute (0,8), $y = \frac{2}{3}x^6 + 4x^2 + 8$

4. $s(t) = t^3 - t^2 + c$, substitute s=12 and t=3, $\boldsymbol{s(t) = t^3 - t^2 - 6}$

Area under a graph

Investigation

1. a) Area 4.5cm^2

c) $\int x\ dx = \frac{1}{2}x^2 + c$

d) $\int_0^3 x\ dx = \left[\frac{1}{2}(3^2) + c\right] - \left[\frac{1}{2}(0^2) + c\right] = 4.5\ cm^2$

Students should notice that the area under the graph between $x = a$ and $x = b$ is equal to $\boldsymbol{\int_a^b y\ dx}$**.**

2. 14

3. -$\frac{1}{4}$