South Sudan

Secondary Mathematics 3

Secondary Mathematics has been written and developed by Ministry of General Education and Instruction, Government of South Sudan in conjunction with Subjects experts. This course book provides a fun and practical approach to the subject of mathematics, and at the same time imparting life long skills to the pupils.

The book comprehensively covers the Secondary 3 syllabus as developed by **Ministry of General Education and Instruction.**

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Secondary Mathematics 3

Student's Book

South Sudan

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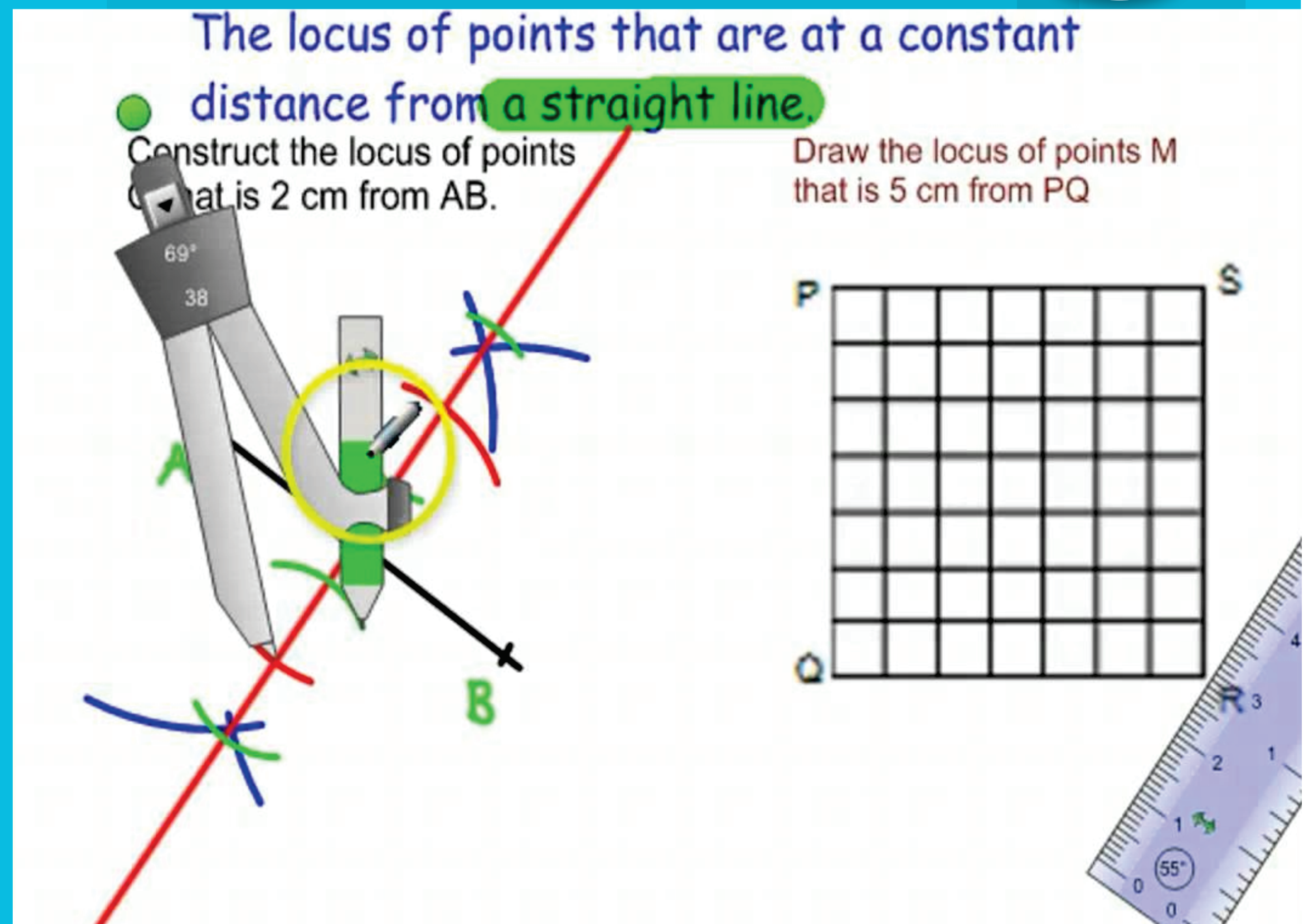

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South Sudan

SECONDARY
3

Mathematics

Student's Book 3

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FOREWORD

I am delighted to present to you this textbook, which is developed by the Ministry of General Education and Instruction based on the new South Sudan National Curriculum. The National Curriculum is a learner-centered curriculum that aims to meet the needs and aspirations of the new nation. In particular, it aims to develop (a) Good citizens; (b) successful lifelong learners; (c) creative, active and productive individuals; and (d) Environmentally responsible members of our society. This textbook, like many others, has been designed to contribute to achievement of these noble aims. It has been revised thoroughly by our Subject Panels, is deemed to be fit for the purpose and has been recommended to me for approval. Therefore, I hereby grant my approval. This textbook shall be used to facilitate learning for learners in all schools of the Republic of South Sudan, except international schools, with effect from 4th February, 2019.

I am deeply grateful to the staff of the Ministry of General Education and Instruction, especially Mr Michael Lopuke Lotyam Longolio, the Undersecretary of the Ministry, the staff of the Curriculum Development Centre, under the supervision of Mr Omot Okony Olok, the Director General for Quality Assurance and Standards, the Subject Panelists, the Curriculum Foundation (UK), under the able leadership of Dr Brian Male, for providing professional guidance throughout the process of the development of National Curriculum and school textbooks for the Republic of South Sudan since 2013. I wish to thank UNICEF South Sudan for managing the project funded by the Global Partnership in Education so well and funding the development of the National Curriculum and the new textbooks. I am equally grateful for the support provided by Mr Tony Calderbank, the former Country Director of the British Council, South Sudan; Sir Richard Arden, Senior Education Advisor of DfID, South Sudan. I thank Longhorn and Mountain Top publishers in Kenya for working closely with the Ministry, the Subject Panels, UNICEF and the Curriculum Foundation UK to write the new textbooks. Finally, I thank the former Ministers of Education, Hon. Joseph Ukel Abango and Hon. Dr John Gai Nyuot Yoh, for supporting me, in my previous role as the Undersecretary of the Ministry, to lead the Technical Committee to develop and complete the consultations on the new National Curriculum Framework by 29 November 2013.

The Ministry of General Education and Instruction, Republic of South Sudan, is most grateful to all these key stakeholders for their overwhelming support to the design and development of this historic South Sudan National Curriculum. This historic reform in South Sudan's education system is intended to benefit the people of South Sudan, especially the children and youth and the future generations. It shall enhance the quality of education in the country to promote peace, justice, liberty and prosperity for all. I urge all Teachers to put this textbook to good use.

May God bless South Sudan. May He help our Teachers to inspire, educate and transform the lives of all the children and youth of South Sudan.

Deng Deng Hoc Yai, (Hon.)
Minister of General Education and Instruction, Republic of South Sudan

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UNIT 1

LOGARITHMS

What is a logarithm?

Logarithms, or "logs", are a way of expressing one number in terms of a "base" number that is raised to some power. Common logs are done with base ten, but some logs ("natural" logs) are done with the constant "e" (2.718 281 828…) as their base. The log of any number is the power to which the base must be raised to give that number.

Consider, $2^3 = 8$, 2 is the base and 3 is the exponent (also called the power or **logarithm**.

So we say that logarithm of 8 to base 2 is 3, and we write this mathematically as

$\log_2 8 = 3$

$2^3 = 8$ is known as the **index notation** while $\log_2 8 = 3$ is the **logarithmic notation**

We are most familiar with base 10 since any number greater than zero can be expressed as 10^x. For example,

$$\log(10) \text{ is } 1 \text{ (because } 10^1 = 10),$$

$$\log(100) \text{ is } 2 \text{ (because } 10^2 = 100), \text{ and}$$

$$\log(2) \text{ is } 0.3 \text{ (because } 10^{0.3} \approx 2).$$

Logs can easily be found for either base on your calculator. Usually there are two different buttons, one saying "log", which is base ten, and one saying "ln", which is a natural log, base e. It is always assumed, unless otherwise stated, that "log" means $\log_{10}$.

Logs are read aloud as "log", "natural log", "ln", or "log base whatever". To read $\log_3 4$, you would simply say "log, base three, of four".

Logs are commonly used in chemistry. The most prominent example is the pH scale. The pH of a solution is the -log([H^+]), where square brackets mean concentration.

In general, if $a^b = c$, then $\log_a c = b$

Being able to change between the index notation and the logarithmic notation helps us simplify logarithmic statements.

Exercise 1: To be done in groups

1. Complete the following index statements and then change them to logarithmic statements

   a) $3^5 =$ ____ b) $3^4 =$ ____

   c) $3^3 =$ ____ d) $3^2 =$ _____

   e) $3^1 =$ _____ f) $3^0 =$ _____

   g) $3^{-1} =$ ______ h) $3^{-3} =$ _____

   i) $9^{\frac{1}{2}} =$ ______ j) $36^{-\frac{1}{2}} =$ ______

2. Hence complete the following statements:

   a) $a^0 =$ _________ therefore $\log_a$ ____ $= 0$

   b) $a^1 =$ _________ therefore $\log_a$ ____ $= 1$

3. Evaluate the following expressions without using a calculator. (a) has been done for you.

   a) $\log_7 49$

   Solution: Let $\log_7 49$ equal x

   $\log_7 49 = x$, therefore $7^x = 49$ (change to index notation)

But $49 = 7^2$

Therefore, $7^x = 7^2$, equating the exponents(powers), we get

$x = 2$

b) $\log_2 64$

c) $\log_9 1$

d) $\log_5 \sqrt{5}$

e) $\log_3 \frac{1}{81}$

f) $\log_{32} 8$

g) $\log_4 4$

4.

a) Find $\log_3(-27)$ by writing it in index notation. Is it possible to get an answer?

b) Evaluate $7^x = -49$. Is it possible to get an answer? Give reasons for your answer.

c) Evaluate $\log_3 0$ by writing it in index notation. What do you notice?

The task above should lead us to following conclusions:

1. $\log_a 1 = 0$

2. $\log_a a = 1$

3. $\log_a a^n = n$

4. $\log_a 0$ has no solution (is undefined)

5. $\log_a(-b)$ has no solution (is undefined)

Laws of logarithms

There are a number of rules known as the laws of logarithms. These allow expressions involving logarithms to be rewritten in a variety of different ways. The laws apply to logarithms of any base but the same base must be used throughout a calculation.

Since a logarithm is simply an exponent, we expect the logarithm laws to work the same as the rules for exponents, and luckily, they do.

The three main laws are stated here:

First Law

$$\log A + \log B = \log AB$$

This law tells us how to add two logarithms together. Adding $\log A$ and $\log B$ results in the logarithm of the product of A and B, that is $\log AB$.

For example, we can write

$$\log_{10} 5 + \log_{10} 4 = \log_{10} (5 \times 4) = \log_{10} 20$$

The same base, in this case 10, is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

Second Law

$$\log A - \log B = \log \frac{A}{B}$$

So, subtracting $\log B$ from $\log A$ results in $\log \frac{A}{B}$.

For example, we can write

$$\log_e 12 - \log_e 2 = \log_e \frac{12}{2} = \log_e 6$$

The same base, in this case e, is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

Third Law

$$\log A^n = n\log A$$

So, for example $\log_{10} 5^3 = 3\log_{10} 5$

You should verify this by evaluating both sides separately on your calculator. Two other important results are

$$\log 1 = 0, \qquad \log_m m = 1$$

The logarithm of 1 to any base is always 0, and the logarithm of a number to the same base is always 1. In particular,

$$\log_{10} 10 = 1, \text{ and } \log_e e = 1$$

Exercise 2

1. Use the first law to simplify the following.
   a) $\log_{10} 6 + \log_{10} 3$,
   b) $\log x + \log y$,
   c) $\log 4x + \log x$,
   d) $\log a + \log b^2 + \log c^3$.
2. Use the second law to simplify the following.
   a) $\log_{10} 6 - \log_{10} 3$,
   b) $\log x - \log y$,
   c) $\log 4x - \log x$.

3. Use the third law to write each of the following in an alternative form.
   a) $3\log_{10} 5$,
   b) $2\log x$,
   c) $\log(4x)^2$,
   d) $5\ln x^4$,
   e) $\ln 1000$.

4. Simplify $3\log x - \log x^2$.

Solving logarithmic equations

There are two major kinds of equations that you will have to solve using logs. In one kind, you will know the log of a number and have to find the number by means raising the base to a power. The other kind gives you the variable in the exponent, and you have to take logs to isolate it. Solving these kinds of problems depends on knowing another property of logs: if the log of a number with an exponent is taken, then the log of that number is multiplied by whatever was in the exponent.

$$\log_c(a^m) = m \log_c(a)$$
$$\log_5(2^3) = 3 \log_5(2)$$

Before we go any further, let's review some definitions that can show the relationship of exponential notation and logarithms.

$$x^0 = 1$$

$$x^a \times x^b = x^{a+b}$$

$$x^a \div x^b = x^{a-b}$$

$$x^{-1} = 1 \div x,\ x^{-2} = 1 \div x^2, \text{ etc.}$$

$$\log_n b = c \text{ means } n^c = b$$

$$\log_{10} 100 = 2 \text{ means } 10^2 = 100$$

Sometimes, you may be required to convert between bases.

So, use these properties to solve the problems below:

Example 1: Solve for x

$$\log_2 x = 8$$

In this problem we can use the definition of logs to help us.
Now we have:

$2^8 = x$

$x = 256$

Example 2: Find the number whose log is 0.25.

Our first step is to set up an equation where x represents the number whose log is 0.25.

$\log x = 0.25$

Now we raise both sides to the power of 10

$10^{\log x} = 10^{0.25}$

$10^{\log x}$ is simply x, so:

$x = 10^{0.25}$

$x = 1.78$

Example 3:

Solve for m

$3^m = 1000$

In this example we can take the log of both sides and eliminate the exponent.

$m \log 3 = \log 1000$

Now we isolate m and solve:

$$m = \frac{\log 1000}{\log 3}$$

$m = 6.29$

Example 4

Solve $log_5(x-2) = 3$

Solution $5^3 = (x-2)$

$125 = (x-2)$

$x = \mathbf{127}$

Solve

$$log_2 x + log_2(x-2) = 3$$

Solution $\boldsymbol{log_2[x(x-2)] = 3}$

$\boldsymbol{2^3 = x(x-2)}$

$8 = x^2 - 2x$

$x^2 - 2x - 8 = 0$ factorize and solve

$(x+2)(x-4) = 0$

$x = -2, x = 4$ we cannot get a logarithm of a negative number, then,

$x = 4$ is the only solution

Exercise 3

In groups, discuss the examples above and answer the following questions in your exercise books

1. Solve for x.

a) $log_9(x-2) = 2$
b) $log_3(2x-1) = 3$
c) $log_{\frac{1}{2}}(3-x) = 5$
d) $log_6(x-5) + log_6 x = 2$
e) $log_2(4x-8) - log_2(x-5) = 4$

Application of logarithms

Logarithms put numbers on a human-friendly scale.

Large numbers break our brains. Millions and trillions are "really big". The trick to overcoming "huge number blindness" is to write numbers in terms of "inputs" (i.e. their power base 10). This smaller scale (0 to 100) is much easier to grasp:

- power of 0 = 10^0 = 1 (single item)
- power of 1 = 10^1 = 10
- power of 3 = 10^3 = thousand
- power of 6 = 10^6 = million
- power of 9 = 10^9 = billion

- power of 12 = 10^{12} = trillion
- power of 23 = 10^{23} = number of molecules in a dozen grams of carbon
- power of 80 = 10^{80} = number of molecules in the universe

A 0 to 80 scale took us from a single item to the number of molecules in the universe.

Logarithms count multiplication as steps

Logarithms describe changes in terms of multiplication: in the examples above, each step is 10 times bigger. With the natural log, each step is "e" (2.71828...) times more.

When dealing with a series of multiplications, logarithms help "count" them, just like addition counts for us when effects are added.

Order of magnitude

In computers, where everything is counted with bits (1 or 0), each bit has a doubling effect. So going from 8 to 16 bits is "8 orders of magnitude" or 2^8= 256 times larger. ("Larger" in this case refers to the amount of memory that can be addressed.) Going from 16 to 32 bits means an extra 16 orders of magnitude, or 2^{16}~ 65,536 times more memory that can be addressed.

Interest Rates

How do we figure out growth rates? A country doesn't intend to grow at 8.56% per year. You look at the GDP one year and the GDP the next, and take the logarithm to find the *implicit* growth rate. Logarithms are how we figure out how fast we're growing.

Measurement Scale: Richter, Decibel, etc.

The idea is to put events which can vary drastically (earthquakes) on a single scale with a small range (typically 1 to 10). Just like PageRank, each 1-point increase is a 10 times improvement in power. The largest human-recorded earthquake was 9.5; the Yucatán Peninsula impact, which likely made the dinosaurs extinct, was 13.

Decibels are similar, though it can be negative. Sounds can go from intensely quiet (pindrop) to extremely loud (airplane) and our brains can process it all. In reality, the sound of an airplane's engine is millions (billions, trillions) of times more powerful than a pindrop, and it's inconvenient to have a scale that goes from 1 to a trillion. Logarithms keep everything on a reasonable scale.

Exercise 4: To be done in groups and presented before the whole class

Research on the following:

a) To what base are the logarithms in mathematical text books written to?

b) How are these logarithms calculated?

Hint: $10^2 = 100$ $therefore$ $log_{10}100 = 2$

Comment on your observations.

UNIT 2

MEASUREMENT AND GEOMETRY

Accuracy: Approximation

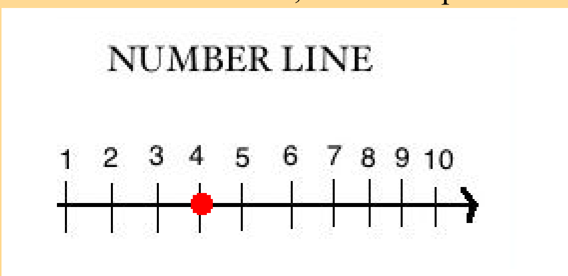
In many instances numbers are not given to exact values and instead they are approximated.

The various ways of approximating are outlined below:

Rounding

Plotting a Number on a Number Line

A **number line** is a line on which numbers are placed in intervals, to show basic numerical calculations. **Plotting** a number on a number line means that you are marking a point designating that number on the number line. Using a basic number line that illustrates numbers 1-10, we would plot the number 4 as shown:



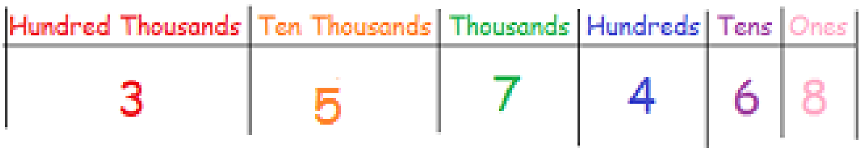
Rounding Using a Number Line

Now let's take our number line knowledge and use that to round whole numbers. Look at our number line from the above example. If you were asked to round the number 6 to the nearest 10, what would you get and where would you plot it? Well, the nearest 10 is going to be a multiple of 10. What are the multiples of 10 that are on opposite ends of the 6? 0 and 10. But is the 6 closer to the 0 or 10? The 10 of course! Therefore, if you were to round 6 to the nearest 10, your answer would be 10 and that is where you would plot it!

Rounding to the Nearest Hundred

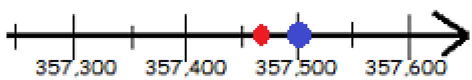
Rounding whole numbers can get a little more complicated when the number is larger. Let's round 357 468 to the nearest hundred.

Step 1 - Find Correct Place Value
Determine which digit is in the hundreds place. It's the 4 because it is the third number from the right and it tells us how many hundreds are in a number.



Place Value

Step 2 - Create Number Line and Plot
Draw a number line with whole numbers as this creates a visual to help you solve the problem. Very important: Because you are rounding to the nearest hundred, your intervals on the number line should go by hundreds. This number line represents the multiples of hundred on either end of 357 468. When we plot the approximate location of 357 468 on the number line (this is the red dot), we can see the nearest hundred. The nearest hundred of 357 468 is 357,500 (plotted with the blue dot).



The red dot plots the actual number. The blue dot plots that number rounded to the nearest hundred.

Let’s say 28 617 people attend a wedding. Imagine you are writing a report and you want to mention how many people attended.

How many would you say to the nearest 10 000? Or to the nearest 1000? Or the nearest 100?

To the nearest 10 000, the figure would be 30 000

To the nearest 1000, the figure would be 29 000

To the nearest 100, the figure would be 28 600

Remember

The same rules for rounding up apply here:

- 5 or more, we 'round up'.
- 4 or less, it stays as it is.

Exercise 1

1. Round the following figures to the nearest 1000:
    a) 68 786
    b) 74 245
    c) 89 000
    d) 4020
    e) 99 500
    f) 999 999

2. Round the following to the nearest 100:
    a) 78540
    b) 6858
    c) 14 099
    d) 950
    e) 2984
    f) 8084

3. Round the following to the nearest 10:
    a) 485
    b) 692
    c) 8874
    d) 4
    e) 997
    f) 83

Decimal Places

Sometimes, rather than rounding off to the nearest whole number, you might need to be a little more accurate. You might need to include some of the digits after the decimal point. In these cases, we can round off the number up to a certain number of decimal places.

Exercise 2

1. Write the following numbers to 1 decimal place.
    a) 5.58
    b) 0.73
    c) 11.86
    d) 12.049
    e) 0.98
    f) 4.04
2. Write the following to 2 decimal places.
    a) 6.473
    b) 9.587
    c) 16.476
    d) 99.996
    e) 3.0037
    f) 9.3048

Significant figures

Sometimes we do not always need to give detailed answers to problems - we just want a rough idea. When we are faced with a long number, we could round it off to the nearest thousand, or nearest million. And when we get a long decimal answer on a calculator, we could round it off to a certain number of decimal places.

Another method of giving an approximated answer is to round off using significant figures.

The word significant means: having meaning.

It was stated in a newspaper that the attendance for a football match was 64 000. But a friend who was attending the same match said the crowd was 64 092. The information from both sources are correct, but it was given to a different degree of accuracy.

64 092 might have been the exact number. When we round off 64 092 to two significant figures, it is 64 000.

The first non-zero digit, reading from left to right in a number, is the first significant figure.

For example, in 64 092, 6 is the first significant figure (s.f.). When we round off 64 092 to four sig. figs, zero in the hundreds place is a significant figure.

Let's round off 64 092 to:
(a) 1 significant figure which is 60 000
(b) 2 significant figures which is 64 000
(c) 3 significant figures which is 64 100
(d) 4 significant figures which is 64 090
(e) 5 significant figures which is 64 092

The accuracy of the answer will depend on the number of significant figures. The answer will be more accurate, if it is given to a higher number of significant figures.

Discussion: In pairs.

The number 43.25 has 4 significant figures.

4 is the most significant because it has the highest value while 5 is the least significant because it has the least value.

What is the value of 4? What is the value of 5?

43.25 to 3 significant figures would become 43.3. Why is this so?

43.25 to 2 significant figures is 43. Why?

Another number 0.00043 approximated to 1 significant figure becomes 0.0004. Explain.

Approximate 0.00852 to 1 significant figure.

Exercise 3

1. Write the following to the number of significant figures in brackets?

   a) 48 599 (1 s.f)

b) 48 599 (3 s.f)
c) 2.578 (3 s.f)
d) 14.952 (3 s.f)
e) 0.0856 (I s.f)
f) 0.00305 (2 s.f)

Upper and lower bounds

Consider the number 4.5, it could have been written to different degrees of accuracy. Since 4.5 is written to 1 dp any number from 4.45 up to but not including 4.55 would be rounded to 4.5.

4.45 is known as the **lower bound (limits)** while 4.55 is known as the **upper bound (limits)**.

Consider 4.50, since it is written to 2 decimal places, only numbers from 4.495 up to but not including 4.505 would be rounded to 4.50. In this case, 4.495 is the **lower bound** while 4.505 is the **upper bound**.

The **absolute error** is given by half the difference between the upper bound and the lower bound. $\boldsymbol{Absolute\ error} = \frac{1}{2}(\boldsymbol{upper\ bound} - \boldsymbol{lower\ bound})$

The **relative error and percentage error** helps us see the magnitude of the error made while measuring

$$\boldsymbol{Relative\ error} = \frac{\boldsymbol{absolute\ error}}{\boldsymbol{actual\ measurement}}$$

$$\boldsymbol{Percentage\ error} = \boldsymbol{relative\ error} \times \mathbf{100\%}$$

Example

The capacity of a jug is stated as 2000 ml to the nearest 10ml. Find,

a) the limits within which the capacity lies.
b) the percentage error.

Solution

a) The upper limit will be 2005ml and the lower limit will be 1995ml

(You can get these values by adding and subtracting half of 10ml to 2000ml)

$$\frac{1}{2} \times 10 = 5$$

$$2000 + 5 = \mathbf{2005}\boldsymbol{ml}$$

$$2000 - 5 = \mathbf{1995}\boldsymbol{ml}$$

b) The absolute error is 5ml

Therefore the relative error is $\frac{5}{2000}$

And the percentage error is $\frac{5}{2000} \times 100 = \mathbf{0.25\%}$

Task

A rectangular card measures 5.3 cm by 2.5 cm. Find

a) the limits within which the length and width lie
b) the area of the card, using the stated values
c) the limits within which the area lies
d) the relative error in the calculation of the area

Exercise 4: Discuss the examples above and then do the following questions in groups.

1. Find the percentage error in the calculation of the volume of a sphere of radius 4.9 cm.
2. Measure the length and width of your blackboard to the nearest whole number. Find the percentage error in the calculation of its area.
3. Measure the radius and height of a soda can to the nearest cm. Find the percentage area in the calculation of the volume.

Use of calculators

In many instances calculations carried out using a calculator produce answers that are not whole numbers. A calculator will give the answer to as many decimal places as will fit its screen. In most cases, this degree of accuracy is neither desirable nor necessary.

Example: Use your calculator to work out $4.64 \div 2.3$, giving your answer to an appropriate degree of accuracy,

Calculator will give 2.0173913, however the answer can be given to 2 d.p. as 2.02.

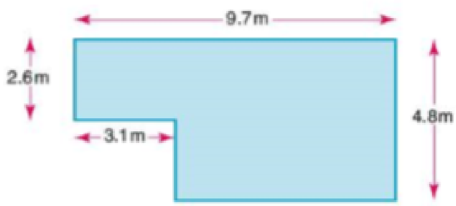
Therefore $4.64 \div 2.3 = 2.02\ (2\ d.p.)$

Exercise 5

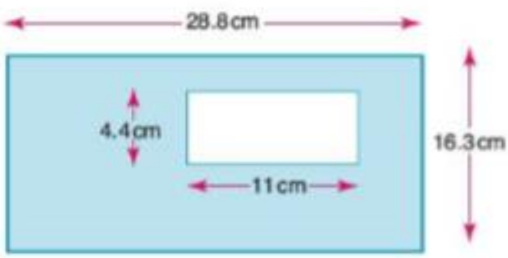
1. Use your calculator to calculate the shaded areas in the following questions. Give your answers to 3 s.f.

a)

b)

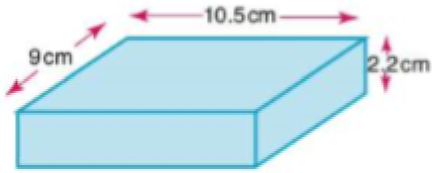


c)

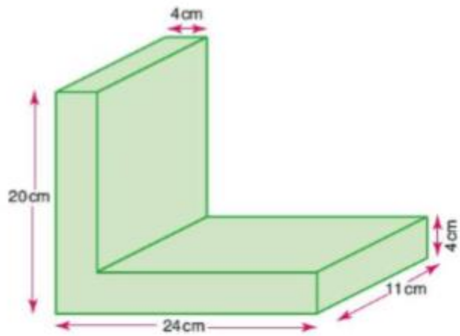


2. Use your calculator to find the surface area and volumes of the following objects to 3 s.f.

a)



b)



3. Measure the dimensions of your desk. Find the volume of material used to make the desk to an appropriate level of accuracy.

LOCUS

A **locus** (plural loci) refers to all the points which fit a particular description. These points can belong to either a region or a line or both. The principal types of loci are explained below.

Activity: To be done as a class.

Find an open area to perform the tasks.

1. **The locus of the points which are at a given distance from a given point**
   Have 1 student in the group tie one end of a rope around his waist and stands stationary. Another student takes the other end and walks around the first student.
   a) Measure the length of the rope.
   b) Describe the path taken by the first student.
   c) Draw a sketch of this path.

2. **The locus of the points which are at a given distance from a given straight line**
    1. Lay a rope on the ground to represent the line.
    2. Have students standing 1m away from each member of the initial group.
    3. Sketch the formation made by the three groups.

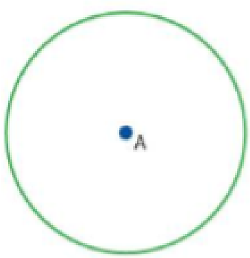
3. **The locus of the points which are equidistant from two given points**
    Put 2 markers 6m apart, have students stand equal distances from the markers. How far from the first 2 students are they? Sketch a diagram showing the path (locus) of points equidistant from 2 given points.

4. **The locus of points equidistant from two given intersecting straight lines**
    Place two ropes on the ground so they cross one another. They should cross at a point. Ask students to stand so they are the same distance (equidistant) from the two lines. What kind of path will be formed? Sketch a diagram showing this path.

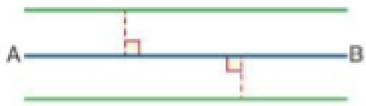
5. **Locus at a fixed distance from a line**
    Your teacher has placed a strip of tape on the classroom floor which forms a segment. The teacher gives each student a yard stick and asks each student to stand exactly 3 feet away from the line on the floor. Can you picture what will happen? If you, and all your classmates stand exactly three feet away from the line, describe where you and your classmates will be standing.

This activity outlines the different types of loci:

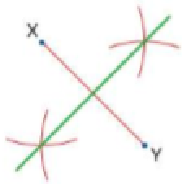
1.



The locus of points equidistant from a point A is the circumference of the circle center A.

2. 

The locus of points which are at a given distance AB from a given point form a set of parallel lines to the given distance AB

3. 

The locus of the points which equidistant from two given points X and Y, is the perpendicular bisector of line XY

4. 

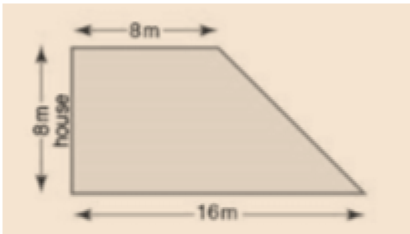
The locus of points equidistant from two given intersecting straight lines is the line bisecting the angle between the two lines.

Application of Loci

Back in class, discuss the following questions in groups.

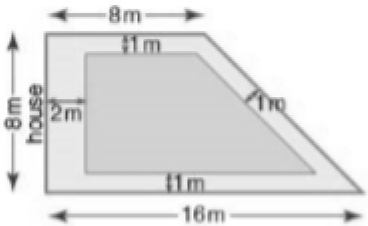
Example

The diagram below shows a trapezoidal garden. Three of its sides are enclosed by a fence and the fourth is adjacent to a house.



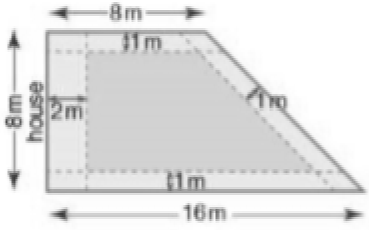
i) Grass is to be planted in the garden. However, it must be at least 2m away from the house and at least 1m away from the fence. Shade the region in which grass **can** be planted

Solution



i) The shaded region will be the locus of all points which are both at least 2m away from the house and at least 1m away from the surrounding fence.

The boundary of the region also forms part of the locus of points.



ii) The shape of the region is the same as in the first case; however, in this instance the boundary is not included in the locus of the points as the grass cannot be exaclty 2m away from the house or exactly 1m away from the fence.

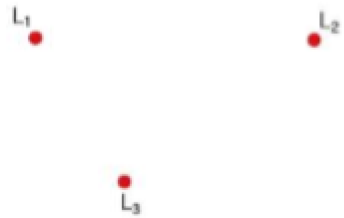
Exercise 6: To be done in groups

1. Draw a scale diagram of a rectangular garden measuring 8m by 6m for each of the following questions.
   a) Draw the locus of all the points at least 1m from the edge of the garden.
   b) Draw the locus of all the points at least 2m from each corner of the garden.
   c) Draw the locus of all the points more than 3m from the center of the garden.
   d) Draw the locus of all the points equidistant from the longer sides of the garden.
2. a) Mark three points L, M and N not in a straight line. By construction find the point which is equidistant from L, M and N.

   b) What would happen if L, M and N were on the same straight line?
3. Draw a line AB 8cm long. What is the locus of a point C such that the angle ACB is always a right angle?
4. Three lionesses L_1, L_2 and L_3 have surrounded a gazelle.

   The three lionesses are equidistant from the gazelle.

   Draw a diagram with the lionesses in similar positions to those shown (left) and by construction determine the position (g) of the gazelle.



5. Three girls are playing hide and seek. Ayshe and Belinda are at the positions shown (left) and are trying to find Cristina. Cristina is on the opposite side of the wall PQ to her two friends.

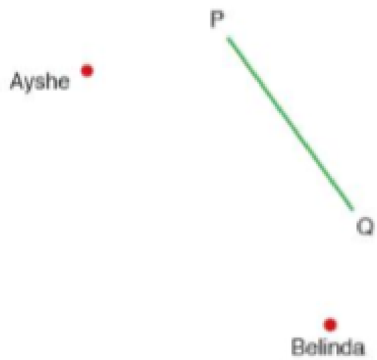
   Assuming Ayshe and Belinda cannot see over the wall identify, by copying the diagram, the locus of points where Cristina could be if:

   a) Cristina can only be seen by Ayshe.

   b) Cristina can only be seen by Belinda.

   c) Cristina cannot be seen by either of her two friends.

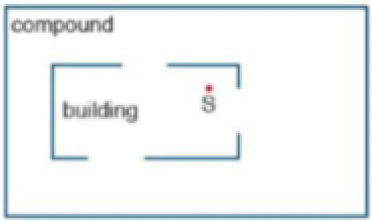
   d) Cristina can be seen by both of her friends.



6. A security guard S is inside a building in the position shown.

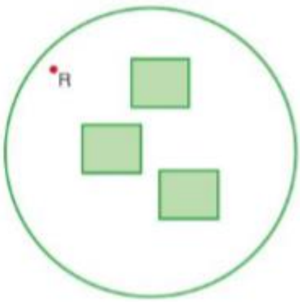
   The building is inside a rectangular compound.

   If the building has three windows as shown, identify the locus of points in the compound which can be seen by the security guard.



7. The circular cage shown (left) houses a snake.

   Inside the cage are three obstacles. A rodent is placed inside the cage at R.
   From where it is lying, the snake can see the rodent. Trace the diagram and identify the regions in which the snake could be lying.



Equation of a circle

Investigation

1. Draw a Cartesian plane using 1cm for 1 unit on both x and y axis. Using a pair of compass, draw a circle center (0, 0) and radius 3 units.
   a) State 2 coordinates on the x −axis where the circle passes through
   b) State 2 coordinates on the y −axis where the circle passes through.
   Mark a point P with x coordinate = 2.
   Read the value of the y coordinate of P.
   Drop a perpendicular from point P to a point N on the x-axis to form a right angled triangle, OPN.
   Using Pythagoras theorem, complete the following equation:

$$\overline{OP}^2 = \overline{ON}^2 + \overline{PN}^2$$

$$3^2 = 2^2 + \_\_\_^2$$

Mark another point Q(x, y) anywhere on the circumference of the circle.

Using Pythagoras theorem, find an equation in terms of x and y.

2. On a different Cartesian plane draw another circle centre (1, 1) and radius 2 units.
   Mark a point P(x, y) on the circumference of the circle. Using Pythagoras theorem, find an equation in terms of x and y.
   Make a summary of what you find and present to the class.

Note: If a computer is available, you could use a graph plotting software, for example, Desmos or Geogebra.

Exercise 7: Work in pairs and show your calculations clearly.

1. Find the equation of a circle center (-1,5) and radius 3 units.
2. Show that the equation of a circle with center (-3, 3) and radius 5 units is given by

$$x^2 + y^2 + 6x - 6y = 7$$

3. Complete the following expressions to make them perfect squares
   a) $x^2 - 6x + \_\_\_\_\_\_ = (x - \_\_\_)^2$
   b) $y^2 + 14y + \_\_\_\_\_\_ = (y + \_\_)^2$

   Using your answer to (a) and (b) above, determine the center and radius of a circle whose equation is $x^2 - 6x + y^2 + 14y + 22 = 0$

4. The equation of a circle is given by the equation $x^2 - 6x + y^2 + 4y - 3 = 0$. Determine the coordinate of the center and the radius of the circle
5. Line AB is the diameter of a circle such that the coordinates of A and B are A (-1, 1) and B (5, 1).
   a) Determine the center and radius of the circle
   b) Hence, find the equation of the circle

UNIT 3

ALGEBRA

Quadratic equations

From previous units, we should know that a **quadratic expression** in x is an expression of the form $ax^2 + bx + c$ where x is the variable and a, b, and c are constants with $a \neq 0$. Examples of quadratic expressions are $\boldsymbol{x^2 + 5x + 6}$ *or* $\boldsymbol{x^2 + 5x}$ *or* $\boldsymbol{x^2}$

A **quadratic equation** is of the form $ax^2 + bx + c = 0$ where $a \neq 0$. Quadratic equations may have two, one or zero real solutions.

Solving quadratic equations

1. **Using factorization (sum and product)**

Example 1

Consider the expansion of $(x+1)(x+6)$:

	x	1
x	x^2	x
6	$6x$	1

$$(x+1)(x+6) = x^2 + x + 6x + 6$$

$$= x^2 + (1+6)x + (1 \times 6)$$

$$= x^2 + (sum\ of\ 1\ and\ 6)x + (product\ of\ 1\ and\ 6)$$

Therefore, $(x+1)(x+6) = x^2 + 7x + 6$

Generally $(x+a)(x+b)=x^2+(a+b)x+ab$

Therefore to factorize x^2+7x+6, we need to look for 2 numbers with a product of 6 and a sum of 7. These numbers are 1 and 6 and so $x^2+7x+6=(x+1)(x+6)$

Example 2

Factorise x^2+5x+6

Look for 2 numbers whose sum is 5 and whose product is 6. These numbers are 3 and 2.

$$x^2+5x+6=(x+2)(x+3)$$

Example 3

Factorise $3x^2+13x+4$

Look for two numbers whose sum is 13 and whose product is 12.

These numbers are 12 and 1

Substitute them back into the expression $3x^2+12x+x+4$

and factorize in pairs $3x^2+12x+x+4=3x(x+4)+1(x+4)$

which becomes $(3x+1)(x+4)$

Example 4

Example

Problem **Solve $(x+4)(x-3)=0$ for x.**

$(x+4)(x-3)=0$		Applying the Principle of Zero Products, you know that if the product is 0, then one or both of the factors has to be 0.
$x+4=0$ or	$x-3=0$	Set each factor equal to 0.
$x+4-4=0-4$	$x-3+3=0+3$	Solve each equation.
$x=-4$ or	$x=3$	

Answer $x=-4$ OR $x=3$

You can check these solutions by substituting each one at a time into the original equation, $(x + 4)(x - 3) = 0$.

Checking $x = -4$	**Checking $x = 3$**	**Trying $x = 5$**
$(x + 4)(x - 3)$	$(x + 4)(x - 3)$	$(x+ 4)(x - 3)$
$= (-4 + 4)(-4 - 3)$	$= (3 + 4)(3 - 3)$	$= (5 + 4)(5 - 2)$
$= (0)(-7) = 0$	$= (7)(0) = 0$	$= (9)(3) = 27 \neq 0$

Example 5

	Example	
Problem	**Solve for a: $5a^2 + 15a = 0$.**	
	$5a^2 + 15a = 0$	Begin by factoring the left side of the equation.
	$5a(a + 3) = 0$	Factor out $5a$, which is a common factor of $5a^2$ and $15a$.
	$5a = 0$ *or* $a + 3 = 0$	Set each factor equal to zero.
	$\frac{5a}{5} = \frac{0}{5}$ *or* $a + 3 - 3 = 0 - 3$ $a = 0$ $\quad a = -3$	Solve each equation.
Answer	$a = 0$ OR $a = -3$	

To check your answers, you can substitute both values directly into the original equation and see if you get a true sentence for each.

Checking $a = 0$	**Checking $a = -3$**
$5a^2 + 15a$	$5a^2 + 15a$
$= 5(0)^2 + 15(0)$	$= 5(-3)^2 + (15)(-3)$
$= 5(0) + 0 = 0$	$= 5(9) - 45 = 0$

Both solutions check.

Exercise 1: To be done in pairs

1. Using the examples above, factorize the following quadratic expressions:

a) $x^2 + 3x + 2$
b) $x^2 - 6x + 8$
c) $x^2 - 8x - 9$
d) $x^2 + (a + b)x + ab$
e) $4x^2 + 12x + 9$
f) $9x^2 + 12x + 4$
g) $5x^2 + 14x - 3$
h) $2x^2 + 3x + 1$
i) $x^2 - 14x + 49$

Solving quadratic equations using factorization

Example 1 – Solve: $x^2 + 16 = 10x$

Step 1: Write the equation in the correct form. In this case, we need to set the equation equal to zero with the terms written in descending order.	$x^2 - 10x + 16 = 0$
Step 2: Use a factoring strategies to factor the problem.	$(x - 2)(x - 8) = 0$
Step 3: Use the Zero Product Property and set each factor containing a variable equal to zero.	$x - 2 = 0$ or $x - 8 = 0$
Step 4: Solve each factor that was set equal to zero by getting the x on one side and the answer on the other side.	$x = 2$ or $x = 8$

Example 2 – Solve: $18x^2 - 3x = 6$

Step 1: Write the equation in the correct form. In this case, we need	$18x^2 - 3x - 6 = 0$

to set the equation equal to zero with the terms written in descending order.	
Step 2: Use a factoring strategies to factor the problem.	$3(6x^2 - x - 2) = 0$ $3(3x - 2)(2x + 1) = 0$
Step 3: Use the Zero Product Property and set each factor containing a variable equal to zero.	$3x - 2 = 0$ or $2x + 1 = 0$
Step 4: Solve each factor that was set equal to zero by getting the x on one side and the answer on the other side.	$3x = 2$ or $2x = -1$ $x = \frac{2}{3}$ $\quad x = -\frac{1}{2}$

Example 3 – Solve: $50x^2 = 72$

Step 1: Write the equation in the correct form. In this case, we need to set the equation equal to zero with the terms written in descending order.	$50x^2 - 72 = 0$
Step 2: Use a factoring strategies to factor the problem.	$2(25x^2 - 36) = 0$ $2(5x + 6)(5x - 6)$
Step 3: Use the Zero Product Property and set each factor containing a variable equal to zero.	$5x + 6 = 0$ or $5x - 6 = 0$
Step 4: Solve each factor that was set equal to zero by getting the x on one side and the answer on the other side.	$5x = -6$ or $x = -1$ $x = -\frac{6}{5}$ $\quad x = \frac{6}{5}$

Example 4 – Solve: $x(2x - 1) = 3$

Step 1: Write the equation in the correct form. In this case, we need to remove all parentheses by distributing and set the equation equal to zero with the terms written in descending order.	$2x^2 - x = 3$ $2x^2 - x - 3 = 0$
Step 2: Use a factoring strategies to factor the problem.	$(2x - 3)(x + 1) = 0$
Step 3: Use the Zero Product Property and set each factor containing a variable equal to zero.	$2x - 3 = 0$ or $x + 1 = 0$
Step 4: Solve each factor that was set equal to zero by getting the x on one side and the answer on the other side.	$2x = 3$ or $x = -1$ $x = \frac{3}{2}$

Example 5 – Solve: $(x + 3)(x - 5) = -7$

Step 1: Write the equation in the correct form. In this case, we need to remove all parentheses by distributing, combine like terms, and set the equation equal to zero with the terms written in descending order.	$x^2 - 5x + 3x - 15 = -7$ $x^2 - 2x - 15 = -7$ $x^2 - 2x - 8 = 0$
Step 2: Use a factoring strategies to factor the problem.	$(x + 2)(x - 4) = 0$
Step 3: Use the Zero Product Property and set each factor containing a variable equal to zero.	$x + 2 = 0$ or $x - 4 = 0$

Step 4: Solve each factor that was set equal to zero by getting the x on one side and the answer on the other side.	$x = -2$ or $x = 4$

Example 6 – Solve: 3x(x + 1) = (2x + 3)(x + 1)

Step 1: Write the equation in the correct form. In this case, we need to remove all parentheses by distributing, combine like terms, and set the equation equal to zero with the terms written in descending order.	$3x^2 + 3x = 2x^2 + 2x + 3x + 3$ $3x^2 + 3x = 2x^2 + 5x + 3$ $x^2 - 2x - 3 = 0$
Step 2: Use a factoring strategies to factor the problem.	$(x + 1)(x - 3) = 0$
Step 3: Use the Zero Product Property and set each factor containing a variable equal to zero.	$x + 1 = 0$ or $x - 3 = 0$
Step 4: Solve each factor that was set equal to zero by getting the x on one side and the answer on the other side.	$x = -1$ or $x = 3$

2. Solve the following equations using factorization.

a) $x^2 - 10x + 25 = 0$

b) $2x^2 + x - 6 = 0$

c) $x^2 - 16 = 0$

d) $x^2 - 14x + 49 = 0$

Perfect squares

A perfect square is a quadratic expression $x^2 + bx + c$ that can be written in the form $(x + a)^2$. For example,

$x^2 - 10x + 25$ is a perfect square because it can be factorized to $(x - 5)^2$

Exercise 2

1. Consider $x^2 + 10x + 25 = (x + 5)^2$ $\quad a = 1, b = 10, c = 25$

   Notice that $\frac{b}{2} = 5$ *and* $\left(\frac{b}{2}\right)^2 = 5^2 = 25$

   Therefore to make $x^2 + 10x$ a perfect square we must add 25.

   a) Give 2 examples of other perfect squares. Factorize these examples and show the relationship between c and $\frac{b}{2}$

   b) What must be added to the following expressions to make them perfect squares?

      i. i) $x^2 + 12x$ ii) $x^2 + 16x$ iii) $x^2 - 8x$ iv) $x^2 + bx$

   c) What must be added to the following expressions to make perfect squares?

      i) $x^2 + \_\_\_\_x + 100$ ii) $x^2 + \_\_\_\_x + 16$ iii) $x^2 + \_\_\_\_x + c$

Solving a quadratic equation by completing the square:

Any quadratic equation can be transformed into a perfect square.

Consider $x^2 + 6x - 11 = 0$, the left side is not a perfect square

Add 10 to both sides to get $x^2 + 6x = 10$

What must be added to the left side to make it a perfect square?

$\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$ Add it on both sides.

$x^2 + 6x + 9 = 11 + 9 = 20$

The left side of the equation is a perfect square and can be factorized

$(x+3)^2 = 20,$

Take the square root of both sides

$\sqrt{(x+3)^2} = \pm\sqrt{20}$

$x + 3 = \pm\sqrt{4 \times 5}$

Subtract 3 from both sides and simplify the surd

$x = -3 \pm 2\sqrt{5}$

$x = -3 + 2\sqrt{5}$ or $x = -3 - 2\sqrt{5}$

Exercise 3

Use the steps above to solve the following using completing the square method.

1. Solve

a) $x^2 + 2x - 7 = 0$
b) $x^2 - 5x + 2 = 0$
c) $x^2 - 8x + 13 = 0$
d) $x^2 - 8x - 30 = 0$

2. When the coefficient of $x^2$is not 1, divide everything by the coefficient as step Solve

a) $2x^2 + 4x + 2 = 0$
b) $2x^2 + 3x - 7 = 0$
c) $3x^2 + 7x - 4 = 0$
d) $2x^2 + 9x + 9 = 0$
e) $ax^2 + bx + c = 0$

Solving quadratic equations using the Quadratic Formula

The last question $ax^2 + bx + c = 0$ gives us the **quadratic formula**

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - (4ac)}}{2a}}$$

When it is not possible to factorise or straightforward to complete the square, the quadratic formula can be used. Great care needs to be taken when substituting in the formula – it is easy to make a mistake.

Example

Solve each of the following equations.

a) $\frac{1}{2}x^2 + x - \frac{1}{10} = 0$

b) $0.04x^2 - 0.23x + 0.09 = 0$

Solution

(a) There are two ways to work this one. We can either leave the fractions in or multiply by the LCD (10 in this case) and solve that equation. Either way will give the same answer. We will only do the fractional case here since that is the point of this problem. You should try the other way to verify that you get the same solution.

In this case here are the values for the quadratic formula as well as the quadratic formula work for this equation.

$$a = \frac{1}{2} \qquad b = 1 \qquad c = -\frac{1}{10}$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\left(\frac{1}{2}\right)\left(-\frac{1}{10}\right)}}{2\left(\frac{1}{2}\right)} = \frac{-1 \pm \sqrt{1 + \frac{1}{5}}}{1} = -1 \pm \sqrt{\frac{6}{5}}$$

In these cases we usually go the extra step of eliminating the square root from the denominator so let's also do that,

$$x=-1\pm\frac{\sqrt{6}}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}=-1\pm\frac{\sqrt{(6)(5)}}{5}=-1\pm\frac{\sqrt{30}}{5}$$

If you do clear the fractions out and run through the quadratic formula then you should get exactly the same result. For the practice you really should try that.

(b) In this case do not get excited about the decimals. The quadratic formula works in exactly the same manner. Here are the values and the quadratic formula work for this problem.

$$a=0.04 \qquad b=-0.23 \qquad c=0.09$$

$$\begin{aligned}x&=\frac{-(-0.23)\pm\sqrt{(-0.23)^2-4(0.04)(0.09)}}{2(0.04)}\\&=\frac{0.23\pm\sqrt{0.0529-0.0144}}{0.08}\\&=\frac{0.23\pm\sqrt{0.0385}}{0.08}\end{aligned}$$

Now, to this will be the one difference between these problems and those with integer or fractional coefficients. When we have decimal coefficients we usually go ahead and figure the two individual numbers. So, let's do that,

$$x=\frac{0.23\pm\sqrt{0.0385}}{0.08}=\frac{0.23\pm 0.19621}{0.08}$$

$$\begin{aligned}x&=\frac{0.23+0.19621}{0.08} &\quad\text{and}\quad& x=\frac{0.23-0.19621}{0.08}\\&=5.327625 &\quad\text{and}\quad& =0.422375\end{aligned}$$

Notice that we did use some rounding on the square root.

Exercise 4: Work in pairs.

2. Choose your method to solve the following equations
   a) $x^2 + 7x + 3 = 0$
   b) $3x^2 - 3x - 2 = 0$
   c) $x^2 - 4x + 3 = 0$
   d) $2x^2 + 11x + 7 = 0$
3. Form a quadratic equation from the information given. Solve either using factorization, completing the square method or the quadratic formula.
   a) A garden is in the shape of a right angled triangle. The lengths of the two shorter sides are $(x+2)m$ and $(4x+4)m$, while the length of the hypotenuse is $(5x)m$.
   b) The length of a rectangle is three times its breadth. If the breadth is decreased by 2m and the length increased by 4m, the area of the rectangle is decreased by a third. Find the length of the original rectangle. Hence, find its area.

Application of quadratics to real life situations

Quadratic equations lend themselves to modeling situations that happen in real life, such as the rise and fall of profits from selling goods, the decrease and increase in the amount of time it takes to run a mile based on your age, and so on.

The wonderful part of having something that can be modeled by a quadratic is that you can easily solve the equation when set equal to zero and predict the patterns in the function values.

The vertex and x-intercepts are especially useful. These intercepts tell you where numbers change from positive to negative or negative to positive, so you know, for instance, where the ground is located in a physics problem or when you'd start making a profit or losing money in a business venture.

The vertex tells you where you can find the absolute maximum or minimum cost, profit, speed, height, time, or whatever you're modelling.

A very common and easy-to-understand application is the height of a ball thrown at the ground off a building. Because gravity will make the ball speed up as it falls, a quadratic equation can be used to estimate its height any time before it hits the ground. *Note: The equation isn't completely accurate, because friction from the air will slow the ball down a little. For our purposes, this is close enough.*

Exercise 5

1. The height of a ball t seconds after it's thrown into the air from the top of a building can be modeled by $h(t) = -16t^2 + 48t + 64$, where $h(t)$ is height in feet. How high is the building, how high does the ball rise before starting to drop downward, and after how many seconds does the ball hit the ground?
2. The profit function telling Lado how much money he will net for producing and selling x specialty umbrellas is given by $P(x) = -0.00405x^2 + 8.15x - 100$.
   What is Lado's loss if he doesn't sell any of the umbrellas he produces, how many umbrellas does he have to sell to break even, and how many does he have to sell to earn the greatest possible profit?
3. Wani ran through a maze in less than a minute the first time he tried. His times got better for a while with each new try, but then his times got worse (he took longer) due to fatigue. The amount of time Wani took to run through the maze on the ath try can be modeled by $T(a) = 0.5a^2 - 9a + 48.5$. How long did Wani take to run the maze the first time, and what was his best time?
4. A highway underpass is parabolic in shape. If the curve of the underpass can be modeled by $h(x) = 50 - 0.02x^2$, where x and $h(x)$ are in feet, then how high is the highest point of the underpass, and how wide is it?

Binomial expansion

A binomial expansion is made by multiplying binomial expressions together. A binomial expression has two terms.

For example $x + y, a + 2b, 2a^2 + a$

The **Binomial Theorem** states that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Note that:

1) The powers of a decreases from n to 0.

2) The powers of b increases from 0 to n.

3) The powers of a and b always add up to n.

Binomial Coefficient

In the expansion of $(a + b)^n$, the $(r + 1)^{\text{th}}$ term is

$$T_{r+1} = \binom{n}{r}a^{n-r}b^r$$

Example 1

Expand

a) $(a + b)^5$

b) $(2 + 3x)^3$

Solution

a) $(a+b)^5 = a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + b^5$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

b) $(2+3x)^3 = 2^3 + \binom{3}{1}2^2(3x) + \binom{3}{2}2(3x)^2 + (3x)^3$

$$= 8 + 36x + 54x^2 + 27x^3$$

Example 2

Find the 7^{th} term of $\left(2 + \dfrac{x^2}{2}\right)^{11}$

Solution

Using the formula $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$T_{6+1} = \binom{11}{6} 2^{11-6} \left(\frac{x^2}{2}\right)^6$$

$$= 462\left(2^5\right)\left(\frac{1}{2}\right)^6 \left(x^2\right)^6$$

$$= 231x^{12}$$

Activity - to be done in groups

1. Expand and simplify the expression $(a+b)^2$

2. Expand and simplify $(a+b)^3$ by first writing it as $(a+b)(a+b)^2$

3. Hence, expand and simplify $(a+b)^4$, $(a+b)^5$ *and* $(a+b)^6$

4. Using your expansions above, fill up the table below:

Expression	Expansion	Coefficients
$(a+b)^0$	1	1
$(a+b)^1$	$a+b$	1 1
$(a+b)^2$	$a^2+2ab+b^2$	1 2 1
$(a+b)^3$		
$(a+b)^4$		
$(a+b)^5$		
$(a+b)^6$		

5. Look at your expansions and note any other patterns you see.

6. Based on your observations, predict what the expansion of $(a+b)^7$ will be.

The Addition Triangle

From the task above activity, notice that the coefficients of the terms of a binomial expansion form a pattern of numbers arranged in the shape of a triangle. This triangle of numbers is often called **Pascal's triangle,** named after a famous European mathematician, Blaise Pascal, despite the triangle's ancient roots in India and China**.**

1

1 1

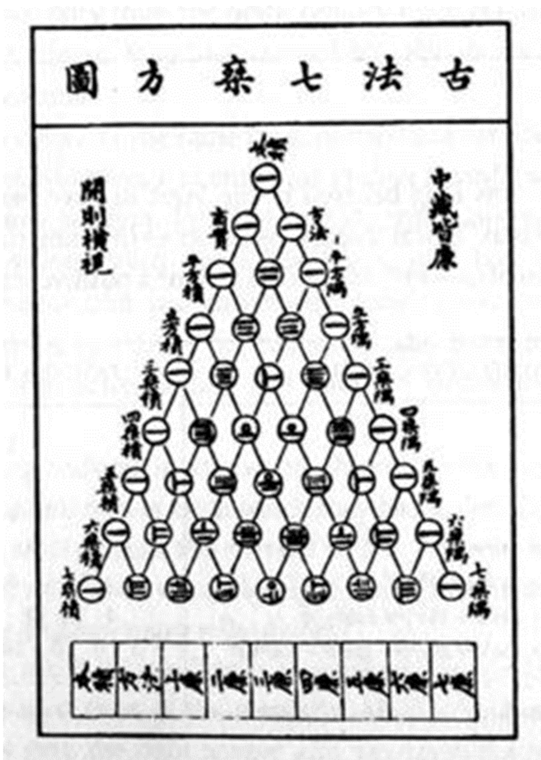
1 2 1

1 3 3 1

1 4 6 4 1

Task

Here is an ancient Chinese copy of the triangle. In groups, decipher the symbols used by the ancient Chinese for numbers.



Exercise 5: to be done in pairs

1. Use the addition triangle to expand

   a) $(a-b)^3$
   b) $(2x+y)^3$
   c) $(1-2b)^4$
   d) $(2x+3y)^4$
   e) $(x-3y)^6$

2. Expand the following giving your answer in surd form

   a) $(1+\sqrt{5})^3$
   b) $(1-\sqrt{5})^4$
   c) $(1+2\sqrt{3})^4$
   d) $(2-3\sqrt{7})^5$

3. Expand each of the following:

   a) $\left(x+\frac{1}{2}\right)^3$
   b) $\left(y+\frac{1}{2}y\right)^4$
   c) $(2x-3)^4$
   d) $\left(3x-\frac{1}{3}y\right)^3$

Investigation

In this investigation your attention to detail and the care with which you present your work is important.

Part A: Binomial Expansions and the Addition Triangle.

1. Expand and simplify each of the following. When finished check with your partner that you have expanded correctly.

(1) $(a+b)^1$ (2) $(a+b)^2$ (3) $(a+b)^3$ (4) $(a+b)^4$.

2. In the triangle of numbers below:

				1					R_0
			1		1				R_1
		1		2		1			R_2
	1		3		3		1		R_3
1		4		6		4		1	R_4

(1) Look for the patterns from row to row and explain how to write down the next row of the triangle.
(2) Make your own copy of the triangle with at least seven rows.
(3) Write down the third row of the triangle. Find and write down the connection between the expansion of $(a+b)^3$ and the third row of the triangle.
(4) Use the results of (3) to expand:

(a) $(a+b)^5$

(b) $(a+b)^6$

(c) $(a+b)^7$

3. Use the Addition triangle to expand any 3 of the following. The first row of this question allows you to demonstrate satisfactory understanding while the second row questions are more challenging and allow you to demonstrate deeper understanding.

(1) $(x+2)^3$ (2) $(x+3)^3$ (3) $(x+5)^3$ (4) $(x-1)^4$

(5) $(2x+3)^3$ (6) $(2x+1)^4$ (7) $(3x-4)^3$ (8) $(2-3x)^4$.

Part B: The Addition Triangle and probability

4. (1) Write down the possibilities when two coins are tossed. How many possibilities are there?

(2) Find the total of row 2 of the triangle.

(3) Find the probability of tossing two coins and obtaining:

(a) No tails

(b) One tail

(c) Two tails

(4) How do these probabilities relate to the triangle? Write a sentence on this.

(5) Write down the possibilities when three coins are tossed. How many possibilities are there?

(6) Find the probability of tossing three coins and obtaining:

(a) No heads.

(b) One head.

(c) Two heads.

(d) Three heads.

(7) Use the triangle to find the following probabilities.

(a) Tossing 4 coins and obtaining 3 tails.

(b) Tossing 6 coins and obtaining 5 heads.

(c) Tossing 6 coins and obtaining 4 tails.

(8) Make up and answer three probability questions of your own that relate to the addition triangle.

Compound proportions, mixtures and rates of work

The proportion involving **two or more quantities** is called **compound proportionality.** The quantities could be directly related or inversely related or both.

Example 1

If one man can do a piece of work in 5 days, how long would it take for 2 men to complete the same piece of work if they are working at the same pace? If one man starts working 1 day before the other how long would it take the two men to complete the work?

Solution

The first man completes $\frac{1}{5}th$ of the work in 1 day and since they both work at the same pace, the second man also completes $\frac{1}{5}th$ of the work in 1 day. Therefore they are able to complete $\frac{1}{5}+\frac{1}{5}=\frac{2}{5}th$ of the work in one day.

They will complete the work in:

$$1 \div \frac{2}{5} = \frac{5}{2} = \mathbf{2\frac{1}{2}}\,\boldsymbol{days}$$

Example 2

1kg of type A sugar costs SSP 125 and 1 kg of type B sugar costs SSP 150. In what proportion should they be mixed so that 1 kg of the mixture costs SSP 140?

Solution

Let x kg of type A sugar be mixed with y kg of type B sugar. Then the cost will be:

$$125x + 150y = 140(x + y)$$

$$= 125x + 150y = 140x + 140y$$

$$-15x = -10y$$

$$\frac{x}{y} = \frac{-10}{-15} = \frac{2}{3}, \text{ Therefore } x:y = 2:3$$

Exercise 6

1. Working 8 hours per day, a glass factory makes 6000 bottles in 3 days. How long would it take to make 10 000 bottles working 9 hours per day?
2. I paid SSP 30 000 for 12 chairs. If my friend Charles want 5 chairs like mine. How much does he have to pay?
3. If 5 trucks transport 120 tons of goods in 2 days: what goods quantity will 7 trucks transport in 3 days?

4. It takes 15 days for a team of 10 workers working 8 hours a day to finish an order. How many people with part time jobs (half the day) will be needed to realize the same work in 10 days?
5. Three gardeners mow the grass of a park in 12 hours. How long will it take if one of them have to go?

Exercise 7: Discuss the following questions in groups and show the correct working

1. Tractor A takes 4 days to mow a field while tractor B takes 7 days to mow the same field. How long will it take both tractors to do the work?
2. Two men each working for 8 hours a day can cultivate an acre of land in 4 days. How long would 6 men, each working 4 hours a day, take to cultivate 4 acres?
3. A shopkeeper buys 1 kg of maize flour costs SSP 100 and 1 kg of millet flour costs SSP 150. In what proportion should they be mixed so that 1 kg of the mixture costs SSP 140? How much should the mixture be sold so that the shopkeeper makes a 10% profit?
4. A cold water tap fills a tub in 10 min and a hot water tap fills the tub in 15 min. How long would they take to fill the tub if both are turned on simultaneously?

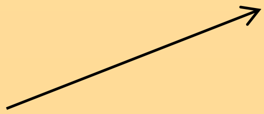
Vectors II

In our day to day life, we come across many queries such as – What is your height? How should a football player hit the ball to give a pass to another player of his team? Observe that a possible answer to the first query may be 1.6 meters, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called scalars.

However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called vectors. In mathematics, physics and engineering, we frequently come across both types of quantities, namely, scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density, resistance etc. and vector quantities like displacement, velocity, acceleration, force, weight, momentum, electric field intensity etc.

The most common notation for vectors is threefold

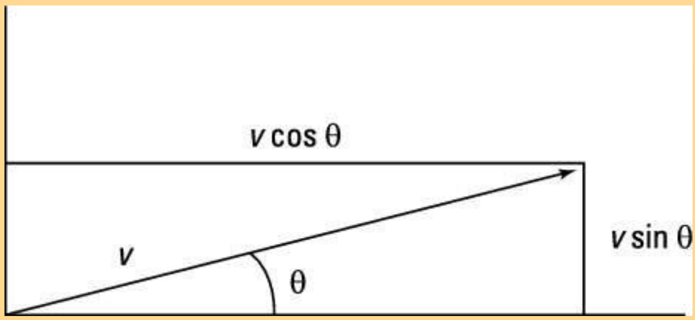
$\overline{AB}$ = **a**



How to find a vector's magnitude and direction

If you're given the components of a vector, such as (3, 4), you can convert it easily to the magnitude and angle way of expressing vectors.

For example, take a look at the vector in the image.



Suppose that you're given the coordinates of the end of the vector and want to find its magnitude, *v*, and angle, theta. Because of your knowledge of trigonometry, you know

$$\frac{y}{x} = \frac{\cancel{v}\sin\theta}{\cancel{v}\cos\theta} = \tan\theta$$

Where tan theta is the tangent of the angle. This means that

$$\theta = tan^{-1}\left(\frac{y}{x}\right)$$

Suppose that the coordinates of the vector are (3, 4). You can find the angle theta as the $\tan^{-1}(4 \div 3) = 53$ degrees.

You can use Pythagoras' theorem to find the magnitude, *v* in the triangle formed by *x, y,* and *v:*

$$v = \sqrt{x^2 + y^2}$$

Plug in the numbers for this example to get

$$v = \sqrt{3^2 + 4^2} = 5$$

So if you have a vector given by the coordinates (3, 4), its magnitude is 5, and its angle is 53 degrees.

Example

Convert the vector $\binom{1}{5}$ into magnitude, angle format.

The correct answer is magnitude 5.1, angle 79 degrees.

Solution

1. Apply Pythagoras' theorem to find the magnitude. Plug in the numbers to get 5.1.
2. $\tan^{-1}(5.0 \div 1.0) = 79$ degrees.

Exercise 8

1. Convert the vector $\binom{5}{7}$ into magnitude, angle form.
2. Convert the vector $\binom{13}{13}$ into magnitude, angle form.
3. Convert the vector $\binom{-1}{1}$ into magnitude, angle form.
4. Convert the vector $\binom{-5}{-7}$ into magnitude, angle form.

Sequences and series

In this section, we will study mathematical patterns. Patterns can help us make predictions about the near and distant future. For example, we can use patterns to:

- predict the population of a country in 20 years
- work out how long it will take to pay off a bank loan
- predict how long a natural resource will last

- calculate the total distance that a bouncing ball will travel
- calculate how long it will take for an investment to double in value.

What is a sequence?

It is a set of numbers which are written in some particular order. For example, take the numbers

$$1, 3, 5, 7, 9, \ldots.$$

Here, we seem to have a rule. We have a sequence of odd numbers. To put this another way, we start with the number 1, which is an odd number, and then each successive number is obtained by adding 2 to give the next odd number.

Here is another sequence:

$$1, 4, 9, 16, 25, \ldots$$

This is the sequence of square numbers. And this sequence,

$$1, -1, 1, -1, 1, -1, \ldots$$

is a sequence of numbers alternating between 1 and -1. In each case, the dots written at the end indicate that we must consider the sequence as an infinite sequence, so that it goes on forever.

On the other hand, we can also have finite sequences. The numbers

$$1, 3, 5, 9$$

form a finite sequence containing just four numbers. The numbers

$$1, 4, 9, 16$$

also form a finite sequence. And so do these, the numbers

$$1, 2, 3, 4, 5, 6, \ldots n.$$

These are the numbers we use for counting, and we have included n of them. Here, the dots indicate that we have not written all the numbers down explicitly. The n after the dots tells us that this is a finite sequence, and that the last number is n.

Here is a sequence that you might recognise:

$$1, 1, 2, 3, 5, 8, \ldots .$$

What do you notice about the sequence?

This is an infinite sequence where each term (from the third term onwards) is obtained by adding together the two previous terms. This is called the Fibonacci sequence.

We often use an algebraic notation for sequences. We might call the first term in a sequence u_1, the second term u_2, and so on. With this same notation, we would write u_n to represent the n-th term in the sequence. So

$$u_1, u_2, u_3, \ldots, u_n$$

would represent a finite sequence containing n terms. As another example, we could use this notation to represent the rule for the Fibonacci sequence. We would write

$$u_n = u_{n-1} + u_{n-2}$$

to say that each term was the sum of the two preceding terms.

Key point:

A sequence is a set of numbers written in a particular order. We sometimes write u_1 for the first term of the sequence, u_2 for the second term, and so on. We write the n^{th} term as u_n.

Exercise 9

1. A sequence is given by the formula $u_n = 3n + 5$, for $n = 1, 2, 3, \ldots$. Write down the first five terms of this sequence.
2. A sequence is given by $u_n = \frac{1}{n^2}$, for $n = 1, 2, 3, \ldots$. Write down the first four terms of this sequence. What is the 10th term?
3. Write down the first eight terms of the Fibonacci sequence defined by $u_n = u_{n-1} + u_{n-2}$, when $u_1 = 1$, and $u_2 = 1$.
4. Write down the first five terms of the sequence given by $n_n = (-1)^{n+1}/n$.

Series

A series is something we obtain from a sequence by adding all the terms together. For example, suppose we have the sequence

$$u_1, u_2, u_3, \ldots, u_n .$$

The series we obtain from this is

$$u_1 + u_2 + u_3 + \ldots + u_n ,$$

and we write S_n for the sum of these n terms. So although the ideas of a 'sequence' and a 'series' are related, there is an important distinction between them. For example, let us consider the sequence of numbers

$$1, 2, 3, 4, 5, 6, \ldots, n.$$

Then $S_1 = 1$, as it is the sum of just the first term on its own. The sum of the first two terms is $S_2 = 1 + 2 = 3$. Continuing, we get

$$S_3 = 1 + 2 + 3 = 6,$$
$$S_4 = 1 + 2 + 3 + 4 = 10,$$

and so on.

Key point

A series is a sequence of the successive sums of the terms in a sequence. If there are n terms in the sequence and we evaluate the sum then we often write S_n for the result, so that

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Exercise 10

Write down $S_1, S_2, \ldots, S_n$ for the sequences

1. 1, 3, 5, 7, 9, 11;
2. 4, 2, 0, −2, -4.

Arithmetic Progression

Consider these two common sequences

$$1, 3, 5, 7, \ldots$$

and

$$0, 10, 20, 30, 40, \ldots .$$

It is easy to see how these sequences are formed. They each start with a particular first term, and then to get successive terms we just add a fixed value to the previous term. In the first sequence we add 2 to get the next term, and in the second sequence we add 10. So the difference between consecutive terms in each sequence is a constant. We could also subtract a constant instead, because that is just the same as adding a negative constant. For example, in the sequence

$$8, 5, 2, -1, -4, \ldots$$

the difference between consecutive terms is -3. Any sequence with this property is called an arithmetic progression, or AP for short.

We can use algebraic notation to represent an arithmetic progression. We shall let a stand for the first term of the sequence, and let d stand for the common difference between successive terms. For example, our first sequence could be written as

$$\begin{array}{cccccc} 1, & 3, & 5, & 7, & 9, & \ldots \\ 1, & 1+2, & 1+2\times 2, & 1+3\times 2, & 1+4\times 2, & \ldots \end{array},$$

and this can be written as

$$a, a+d, a+2d, a+3d, a+4d, \ldots$$

where $a = 1$ is the first term, and $d = 2$ is the common difference.

Investigation: To be done in groups.

1. Consider the sequence 8, 11, 14, 17 …

$$a = 8 \text{ and } d = 3$$

a) Write the second term in terms of the first term
b) Write the third term in terms of the first term
c) Write the fourth term in terms of the first term
What do you notice?
d) Using a and d write an expression for the nth term.
Verify your formula by using it to get the 4^{th} term.
e) Justify your formula by using it to get the 10^{th} term of the sequence 3, 5, 7, 9…

From your investigation, it can be stated that the n^{th} term of any arithmetic sequence is given by

$$\boldsymbol{n^{th}\ term = a + (n-1)d}$$

We also sometimes write ℓ for the last term of a finite sequence, $\ell = a + (n-1)d$.

Exercise 11

1. Write down the first five terms of the AP with first term 8 and common difference 7.
2. Write down the first five terms of the AP with first term 2 and common difference -5.
3. What is the common difference of the AP $11, -1, -13, -25...?$
4. Find the 17th term of the arithmetic progression with first term 5 and common difference 2.
5. Write down the 10th and 19th terms of the AP:
   i) 8, 11, 14, …
   ii) 8,5,2....
6. An AP is given by $k, 2\frac{k}{3}, \frac{k}{3}, 0, \ldots.$
   (i) Find the sixth term.
   (ii) Find the n th term.
   (iii) If the 20th term is equal to 15, find k.
7. For each sequence:
   i) Find the 15th term.
   ii) Find an expression for the nth term.
      a) 3, 6, 9…
      b) 25, 40, 55…

c) 36, 41, 46 …

d) 100, 87, 74 …

8. Find the number of terms in each sequence.

a) 5, 10, 15… 255

b) 4.8, 5.0, 5.2… 38.4

c) $\frac{1}{2}, \frac{7}{8}, \frac{5}{4}, \dots \dots \dots 14$

d) 250, 221, 192… -156

9. An arithmetic progression has first term 19 and 15th term 31.6. Find the common difference.

10. The second term of an arithmetic sequence is 7 and the ninth term is 28. Find the common difference and the first term of the sequence.

The sum of an arithmetic series

Sometimes we want to add the terms of a sequence. What would we get if we wanted to add the first n terms of an arithmetic progression? We would get

$$Sn = a + (a + d) + (a + 2d) + \dots + (\ell - 2d) + (\ell - d) + \ell.$$

Investigation

1. Consider the series $S_n = 3 + 5 + 7 + 9 + 11 + \cdots + n$

   $S_4 = 3 + 5 + 7 + 9 + 11$ equation 1

   This can also be written as

   $S_4 = 11 + 9 + 7 + 5 + 3$ equation 2

   a) Add equations 1 and 2, use your new equation to find S_4

   b) Use the same process to find the sum of the first 10 terms of the sequence.

2. Consider the sequence $a, a + d, a + 2d \dots \dots \dots .., a + (n - 1)d$. Using the same process, find an expression for S_n in terms of a and n.

3. Form your own arithmetic series and find the sum of the first 10 terms.

From the investigation above:

The sum of the first n terms of an arithmetic series can be found using the formula:

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

As the last term, $\ell = a + (n-1)d$, we can also write

$$S_n = \frac{n}{2}(a + l)$$

Example 1

Find the sum of the first 50 terms of the sequence

1, 3, 5, 7, 9, ...

Solution

This is an arithmetic progression, and we can write down

$a = 1, \qquad d = 2, \qquad n = 50.$

We now use the formula, so that

$$\begin{aligned} Sn &= \tfrac{1}{2}n(2a + (n-1)d) \\ S_{50} &= \tfrac{1}{2} \times 50 \times (2 \times 1 + (50-1) \times 2) \\ &= 25 \times (2 + 49 \times 2) \\ &= 25 \times (2 + 98) \\ &= 2500 \end{aligned}$$

Example 2

Find the sum of the series

$1 + 3{\cdot}5 + 6 + 8{\cdot}5 + \ldots + 101.$

Solution

This is an arithmetic series, because the difference between the terms is a constant value, 2·5. We also know that the first term is 1, and the last term is 101. But we do not know how many terms are in the series. So we will need to use the formula for the last term of an arithmetic progression,

$$\ell = a + (n-1)d$$

to give us

$$101 = 1 + (n-1) \times 2{\cdot}5.$$

Now this is just an equation for n, the number of terms in the series, and we can solve it. If we subtract 1 from each side we get

$$100 = (n - 1) \times 2{\cdot}5$$

and then dividing both sides by 2·5 gives us

$$40 = n - 1$$

so that $n = 41$. Now we can use the formula for the sum of an arithmetic progression, in the version using ℓ, to give us

$$\begin{aligned} S_n &= \tfrac{1}{2}n(a + \ell) \\ S_{41} &= \tfrac{1}{2} \times 41 \times (1 + 101) \\ &= \tfrac{1}{2} \times 41 \times 102 \\ &= 41 \times 51 \\ &= 2091. \end{aligned}$$

Example 3

An arithmetic progression has 3 as its first term. Also, the sum of the first 8 terms is twice the sum of the first 5 terms. Find the common difference.

Solution

We are given that $a = 3$. We are also given some information about the sums S_8 and S_5, and we want to find the common difference. So we shall use the formula

$$S_n = \tfrac{1}{2}n(2a + (n - 1)d)$$

for the sum of the first n terms. This tells us that

$$S_8 = \tfrac{1}{2} \times 8 \times (6 + 7d).$$

and that

$$S_5 = \tfrac{1}{2} \times 5 \times (6 + 4d)$$

So, using the given fact that $S_8 = 2S_5$, we see that

$$\begin{aligned} \tfrac{1}{2} \times 8 \times (6 + 7d) &= 2 \times \tfrac{1}{2} \times 5 \times (6 + 4d) \\ 4 \times (6 + 7d) &= 5 \times (6 + 4d) \\ 24 + 28d &= 30 + 20d \\ 8d &= 6 \\ d &= \frac{3}{4} \end{aligned}$$

Discussion questions: To be done in groups for presentation to the whole class

1. The sum of the first 10 terms of an arithmetic series is 400. If the sum of the first 6 terms of the same series is 120, find the 15^{th} term.
2. The sum of the first 20 terms of an arithmetic series is 7.5. If the third term of the series is 2, find the sum of the first 13 terms.
3. In January of a certain year, a new hotel is able to sell 500 cups of tea. In February, they sell 600 cups of tea and in March they sell 700 cups and so on in an arithmetic sequence.
   a) How many cups of tea will they expect to sell in December of that year?
   b) Calculate the **total** number of cups of tea they expect to sell by the end of that year.

Exercise 12

1. Find the sum of the first 23 terms of the AP $4, -3, -10 \ldots$
2. An arithmetic series has first term 4 and common difference $\frac{1}{2}$. Find
   (i) the sum of the first 20 terms,
   (ii) the sum of the first 100 terms.
3. Find the sum of the arithmetic series with first term 1, common difference 3, and last term 100.
4. The sum of the first 20 terms of an arithmetic series is identical to the sum of the first 22 terms. If the common difference is -2, find the first term.

Investigation: Saving money

Joel decides to start saving money. He saves SSP 20 the first week, SSP 25 the second week, SSP 30 the third week, and so on.

i. Copy and complete the table below in an Ms Excel spreadsheet to show how much Joel saves each week, and how much he has saved in total, for the first 8 weeks.

Week number	Weekly savings	Total savings
1	20	20

2	25	45
3	30	75
4		
5		
9		
7		
8		

ii. How much will Joel save in the 10th week? In the 17th week?
iii. How much money will Joel save in total in the first year?
iv. How long will it take for him to save a total of at least SSP 1000?
v. Try to write a formula for the amount of money Joel saves each week. Let M represent the amount of money he saves each week, and let n represent the week number.
vi. Try to write a formula for the total amount of money Joel has saved. Let T represent his total savings, and let n represent the number of weeks.

Geometric progression

We shall now move on to the other type of sequence we want to explore. Consider the sequence

$$2, 6, 18, 54, \dots .$$

Here, each term in the sequence is 3 times the previous term. And in the sequence

$$1, -2, 4, -8, \dots ,$$

each term is −2 times the previous term. Sequences such as these are called geometric progressions, or GPs for short.

Example 1

Determine first 6 members of a geometric progression if stands a3 = 8 and a7 = 128.

Solution

$$a_r = a_s q^{r-s}$$
$$a_7 = a_3 q^4$$
$$q^4 = \frac{a_7}{a_3}$$
$$q = \sqrt[4]{\frac{a_7}{a_3}}$$
$$q = \sqrt[4]{\frac{128}{8}}$$
$$q = \sqrt[4]{16}$$
$$q = 2 \quad \vee \quad q = -2$$

$$a_3 = a_1 q^2$$
$$a_1 = \frac{a_3}{q^2} = \frac{8}{2^2} = \frac{8}{4} = 2 \quad \vee \quad a_1 = \frac{a_3}{q^2} = \frac{8}{(-2)^2} = \frac{8}{4} = 2$$
$$a_1 = 2$$

$$\{a_n\}^6{}_{n=1} = 2;4;8;16;32;64$$
$$\{a_n\}^6{}_{n=1} = 2;-4;8;-16;32;-64$$

Example 2

If a number is added to 2, 16 and 58, it results in first 3 geometric progression members. Find out the number and enumerate first 6 members of the progression.

Solution

$$x^2 - 66x + 128 = 0$$
$$(x-2)(x-64) = 0$$
$$x_1 = 2 \Rightarrow a_1 = 2$$
$$x_2 = 64 \Rightarrow a_6 = 64$$

$$a_6 = a_1 q^5$$
$$q = \sqrt[5]{\frac{a_6}{a_1}}$$
$$q = \sqrt[5]{\frac{64}{2}}$$
$$q = \sqrt[5]{32}$$
$$q = 2$$

$$(a_n)^6{}_{n=1} = 2;4;8;16;32;64$$

Example 3

A workman agreed to work under following conditions: His salary for the first day of work will be SSP 1, for the second day of work SSP 2, for the third day of work SSP 4, and so on. How long does he have to work to earn SSP 4095?

Solution

$$a_1 = 1, q = 2, S_n = 4095, n = ?$$

$$S_n = \frac{a_1(q^n - 1)}{q - 1}$$
$$4095 = \frac{1.(2^n - 1)}{2 - 1}$$
$$4095 = 2^n - 1$$
$$2^n = 4096$$
$$2^n = 2^{12}$$
$$n = 12$$

The workman needs to work for 12 days.

Investigation: To be done in groups

Consider the sequence: 1, 5, 25, 125,

a) What is the value of a *and* r?
b) How can we get the second term from the first term?
c) How can we get the third term from the first term?
d) Write an expression for the n^{th} term using a and r.
e) Use your expression to find the 10^{th} term of the sequence.
f) Does our expression work for all geometric sequences? Prove your formula by generating your own geometric sequence and using your formula to get the 5^{th} term.

From your investigation, the n^{th} term of a geometric sequence is given by

$$\boldsymbol{n^{th}\ term = ar^{n-1}}$$

Exercise 13

1. Write down the first five terms of the geometric progression which has first term 1 and common ratio $\frac{1}{2}$.
2. Find the 10th and 20th terms of the GP with first term 3 and common ratio 2.
3. Find the 7th term of the GP $2, -6, 18, \ldots,$

Investigation

1. Consider the series $2 + 6 + 18 + 54 + \cdots \ldots \ldots$

a) What is the value of first term and the common ratio?
b) Find the sum of the first 4 terms $2 + 6 + 18 + 54 =$
c) Let $S_4 = 2 + 6 + 18 + 54$……………equation 1
   Multiply S_4 by the common ratio, let this be equation 2
d) Subtract equation 1 from equation 2, and find S_4. Is it the same value you obtained in (b)?

2. Consider the series $a, ar, ar^2, ar^3, \ldots \ldots ..ar^{n-1}$

a) Let $S_n = a + ar + ar^2 + \cdots \ldots \ldots + ar^{n-1}$
   i. Use the steps (c) and (d) to find an expression for S_n

b) Use your expression for S_n to find the sum of the first 4 terms of the series 8+24+72+……….

From your investigation,

the sum of a geometric series is given by $S_n = \frac{a(r^n-1)}{r-1}$

or $S_n = \frac{a(1-r^n)}{1-r}$ **where** $r \neq 1$

Exercise 14

1. The first term of a geometric series is 4. What is the sum of the first 6 terms of the series if the common ration is 3?
2. In a geometric series the first term is 2 and the common ratio is 3. Find the number of terms that will give a sum of 242.
3. The fourth term of a geometric sequence is 192. If the first term of the sequence is 3, find
    a) the common ratio
    b) the number of terms that will give a sum of 255
4. John's salary is SSP 15 000. Every year his salary increases by 10% of the amount in the previous year. Find the total amount he will have earned in 5 years.

Matrices and transformations

A matrix is a rectangular arrangement of numbers. Each entry in the matrix is called an element. Matrices are classified by the number of rows and the number of columns that they have, a matrix **A** with m rows and n columns is an $m \times n$ (said $'m$ by n') matrix, and this is called the **order** of **A**

Transformations change the shape, size and position of objects. Examples of transformations are reflection, rotation, enlargement and translation.

Example

Given

$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & -1 & 0 \end{bmatrix}$, Then **A** has order 2×3 (rows first, columns second).The elements of **A** can be denoted by a_{ij}, being the element in the ith row and jth column of **A**. In the above case, $a_{11} = 1, a_{23} = 0, etc.$

Investigation

1. Draw triangle PQR with P(1, 2), Q(3, 2) and R(3, 5).
   a) Write down the position vector of points P, Q, R. The first one has been done for you.
      a. $\widetilde{OP} = \begin{pmatrix}1\\2\end{pmatrix}$
   b) Consider the 2×2 matrix $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$, pre-multiply each position vector by this matrix, to get points P', Q' and R'.
      The first one has been done for you:
      a. $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$, this gives us a new point P' (-1, 2)
   c) On the same Cartesian plane mark points P', Q' and R' and join to form triangle P'Q'R".
   d) Is triangle P'Q'R' a transformation of triangle PQR? What is the transformation? Describe it fully.

2. Draw triangle ABC with A(1, 1) B(3, 1) and C(3, 4).
   a) Pre-multiply the position vectors of A, B and C by the matrix $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ to get A′, B′ and C′.
   b) Draw triangle A′B′C′ on the same axes and compare it to triangle ABC.
   c) Is there a transformation? If yes, describe it fully.

3. On a separate grid draw a triangle of your own choice and label it.
   a) Pre multiply the position vectors of your vertices with the matrix $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$
   b) Is there a transformation? Describe it fully.

4. a) On a separate grid, draw a rectangle of your choice and label it.
   b) Follow steps (b) and (c) above using matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

Summarize your results.

Finding the matrix of transformation

Computer engineers perform matrix transformations in the computer animation used in movies and video games. The animation models use matrices to describe the locations of specific points in images. Transformations are added to images to make them look more realistic and interesting.

Matrix transformations: A transformation is a function which maps the points of a set X, called the pre-image, onto a set of points Y, called the image, or onto itself. A transformation is a change of position of points, lines, curves or shapes in a plane, or a change in shape due to an enlargement or reduction by a scale factor. Each point of the plane is transformed or mapped onto another point.
The transformation, T, is written as:

$$T:\begin{bmatrix}x\\y\end{bmatrix}\to\begin{bmatrix}x'\\y'\end{bmatrix}$$

which means T maps the points of the original or the pre-image point (x, y) onto a new position point known as the image point (x', y') .

Any transformation that can be represented by a 2 × 2 matrix, $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ is called a linear transformation. The origin never moves under a linear transformation.
An invariant point or fixed point is a point of the domain of the function which is mapped onto itself after a transformation. The pre-image point is the same as the image point.

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$$

An invariant point of a transformation is a point which is unchanged by the transformation. For example, a reflection in the line $y = x$ leaves every point on the line $y = x$ unchanged. The transformations that will be studied in this chapter are translations, reflections, rotations and enlargement.

Example
Use the unit square to describe the transformation given by

the matrix $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.

Multiply each point of the unit square:

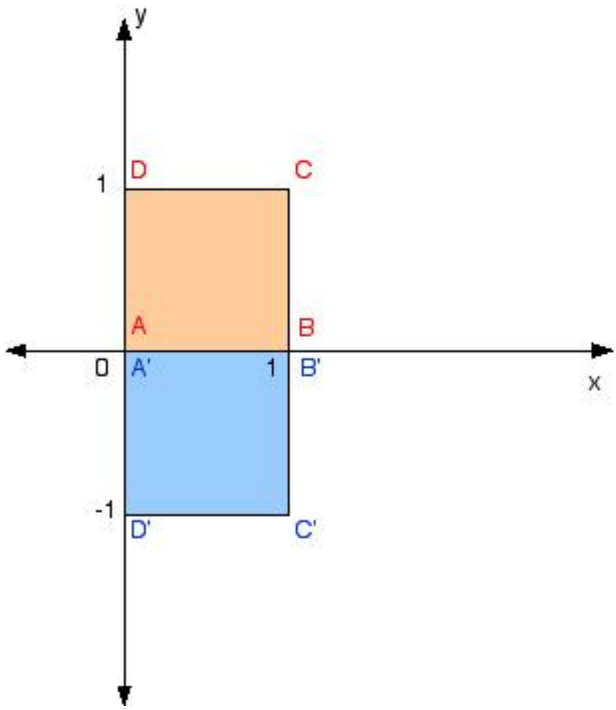
$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\underset{A}{\begin{pmatrix}0\\0\end{pmatrix}}=\underset{A'}{\begin{pmatrix}0\\0\end{pmatrix}}$$

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\underset{B}{\begin{pmatrix}1\\0\end{pmatrix}}=\underset{B'}{\begin{pmatrix}1\\0\end{pmatrix}}$$

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\underset{C}{\begin{pmatrix}1\\1\end{pmatrix}}=\underset{C'}{\begin{pmatrix}1\\-1\end{pmatrix}}$$

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\underset{D}{\begin{pmatrix}0\\1\end{pmatrix}}=\underset{D'}{\begin{pmatrix}0\\-1\end{pmatrix}}$$

Diagram



From the diagram it can be seen that the unit square has been reflected in the x-axis.

ALL points in the plane would be **reflected in the *x*-axis** by the matrix $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

Note that the origin (0, 0) always remains invariant (unchanged) under a 2×2 transformation matrix.

Activity: Complete in pairs.

1. Draw triangle ABC with A(1, 3) B(3, 3) and C(2, 5) and its image under a certain transformation,

   $A'(1, -3), B'(3, -3)$ and $C'(2, -5)$

a) Let the matrix be represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Pre-multiply each of the vertices of the triangle and equate to the corresponding image. The first one has been done for you.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

This will result in : $\quad a + 3b = 1$

$$c + 3d = -3$$

b) From the 6 equations obtained, solve the ones with like terms simultaneously.

c) State the matrix.

Inverse of a transformation

If **A** is a transformation that maps object C onto C', then a transformation that maps C' back onto C is called **the inverse of transformation A** written as $\boldsymbol{A^{-1}}$.

Note that A multiplied by $\boldsymbol{A^{-1}}$ gives the identity matrix.

$$\boldsymbol{A^{-1}A = I}$$

Isometric transformations

A transformation that does not alter the size and shape of an object is called an isometry. Examples of isometric transformations are reflection, rotation and translation.

Non-isometric transformations:

These are transformations in which the size and/or shape of the object is different from the size and shape of the image. An example is enlargement in which the object and image are similar.

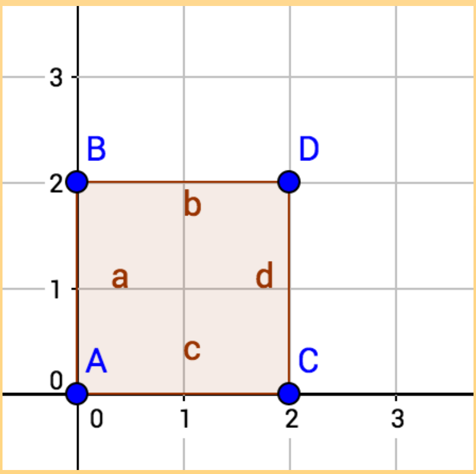
Other examples are Shear and Stretch

Shear

A shear is a transformation in which all points along a given line L remain fixed while other points are shifted parallel to L by a distance proportional to their perpendicular distance from L. Shearing a plane figure does not change its area.

A shearing transformation rotates one axis so that the x-axis and y-axis are no longer perpendicular. The coordinates of the node are shifted by the specified multipliers.

Example



You have a square with vertices at (0, 0) (2, 0) (0, 2) and (2, 2). Shear the square with x-axis invariant, displacement 3 and scale factor 1.5.

x-axis invariant shear means we used this matrix:

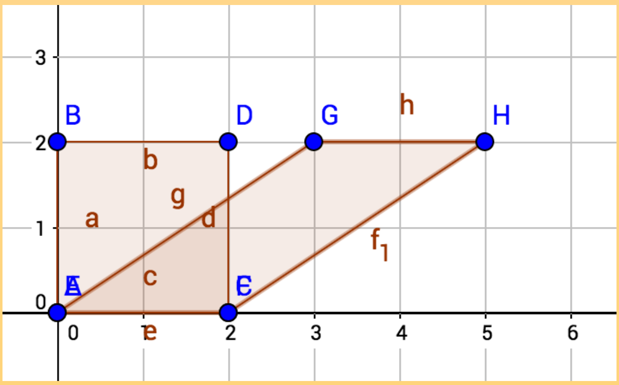
Scale factor 1.5 means our matrix is: $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

So our matrix multiplication is: $\begin{pmatrix} 1 & 1.5 \\ 0 & 1 \end{pmatrix}$

Multiplications: $\begin{pmatrix} 1 & 1.5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

$$1 \times 0 + 1.5 \times 0 = 0$$
$$1 \times 2 + 1.5 \times 0 = 2$$
$$1 \times 0 + 1.5 \times 2 = 3$$
$$1 \times 2 + 1.5 \times 2 = 5$$
$$0 \times 0 + 1 \times 0 = 0$$
$$0 \times 2 + 1 \times 0 = 0$$
$$0 \times 0 + 1 \times 2 = 2$$
$$0 \times 2 + 1 \times 2 = 2$$

So the coordinates of the sheared square are: $\begin{pmatrix} 0 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \end{pmatrix}$



Activity: Complete in pairs.

Consider the triangle ABC where A(0, 0), B(2, 0) and C(1, 1) and its image A'B'C' where A'(0,0), B'(2,0) and C'(3,1).

1. a) draw ABC and $A;B'C'$ on squared paper or graph paper.

b) Can you describe what has happened to the object?

Points A and B have remained the same while point C has moved. Both points A and B are on the x-axis, so we say that the x-axis is **invariant** (any point on the x-axis does not move). We say that this transformation is a **shear, x- axis invariant and shear factor of 2.**

2. a) Using the same triangle ABC as the object, draw its image A'B'C' where $A'(0,0)$, $B'(2,0)$ and $C'(-1,1)$

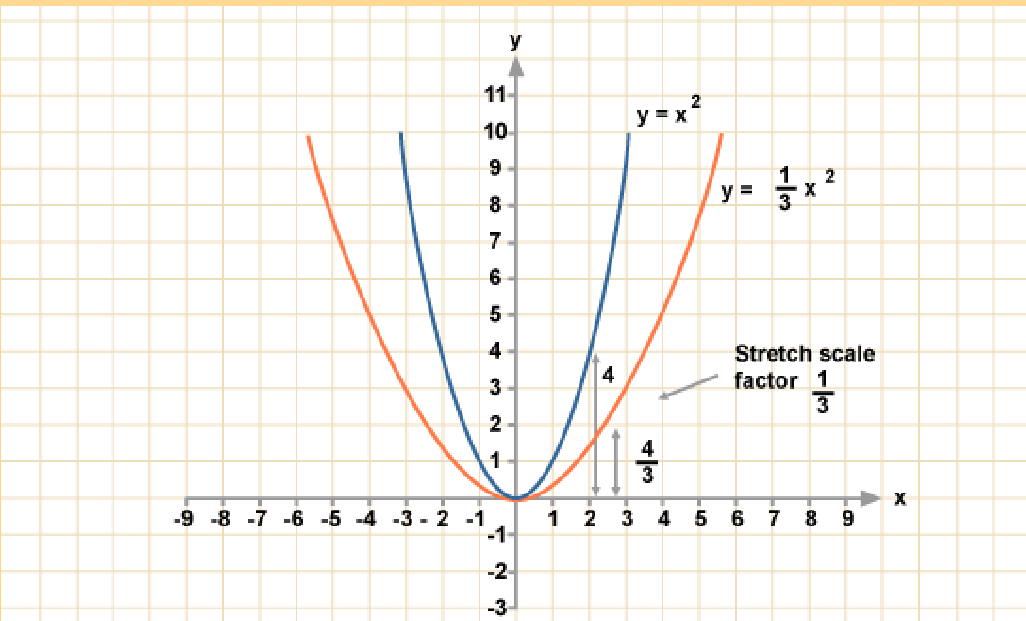
   b) What is the shear factor of this transformation? Which is the invariant line?

3. a) Using the same triangle ABC, draw its image $A'(0,0)$, $B'(2,4)$ ad $C'(1,3)$

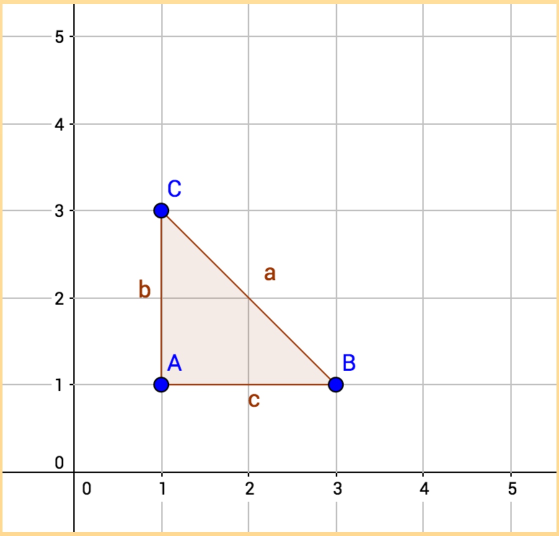
   b) What is the shear factor of this transformation? Which is the invariant line?

Stretch

A stretch, also known as a one-way enlargement, is defined parallel to a specified or direction. Any line parallel to this direction is invariant, and there will be one line of invariant points perpendicular to this direction. Points of the plane are moved so that their distances from the line of invariant points are increased by a factor of k. Distances are, in general, not preserved and areas are increased by a factor of k.



Example



You have a triangle with vertices at (1, 1) (3, 1) and (1, 3). Find the matrix for a stretch, factor 3, x-axis invariant.

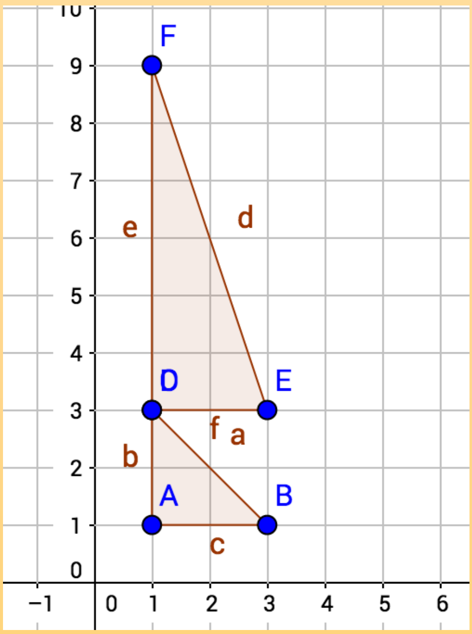
Stretch means we are look at the top half of the table, and then x-axis invariant means $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

Our stretch factor, k, is 3 so our matrix is: $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

So our matrix multiplication will be: $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

$$1 \times 0 + 0 \times 1 = 1$$
$$1 \times 3 + 0 \times 1 = 3$$
$$1 \times 1 + 0 \times 3 = 1$$
$$0 \times 1 + 3 \times 1 = 3$$
$$0 \times 3 + 3 \times 1 = 3$$
$$0 \times 1 + 3 \times 3 = 0$$

The matrix representing the stretch - so the coordinates of the stretched triangle are $\begin{pmatrix} 1 & 3 & 1 \\ 3 & 3 & 9 \end{pmatrix}$.



Activity: Complete in pairs.

1. Given rectangle PQRS, P(2, 3). Q(4, 3), R(4, 0) S(2, 0) and its image P'(2, 6) Q'(4, 6), R'(4,0) and R'(2, 0). Draw the object and image on squared paper or graph paper. What do you notice? What is the equation of the invariant line? How can we find the stretch factor?
2. Using the same rectangle, draw the image of PQRS after a stretch scale factor 2, y-axis invariant.
3. Draw your own shape on the Cartesian plane, find its image after

   i) a shear scale factor 2, x-axis invariant

   ii) a stretch scale factor 2, y-axis invariant

   Present answers to the whole class.

UNIT 4:

CALCULUS

Differentiation

In day to day life we are often interested in the extent to which a change in one quantity affects a change in another related quantity. This is called a *rate of change*. For example, if you own a motor car you might be interested in how much a change in the amount of fuel used affects how far you have travelled. This rate of change is called *fuel consumption*. If your car has high fuel consumption then a large change in the amount of fuel in your tank is accompanied by a small change in the distance you have travelled. Sprinters are interested in how a change in time is related to a change in their position. This rate of change is called *velocity*. Other rates of change may not have special names like fuel consumption or velocity, but are nonetheless important. For example, an agronomist might be interested in the extent to which a change in the amount of fertiliser used on a particular crop affects the yield of the crop. Economists want to know how a change in the price of a product affects the demand for that product.

Differential calculus is about describing in a precise fashion the ways in which related quantities change.

Gradients of secant lines and tangent lines:

Investigation

1. On graph paper draw the graph of $y = x^2 + 1$ for $-2 \leq x \leq 2$. Use a scale of 2cm for 1 unit on the x-axis and and y-axis.
   a) Mark points A at $x = -1.5$, B at $x = -1$, C at $x = -0.5$, P at $x = 0$, D at $x = 0.5$, E at $x = 1$ and F at $x = 1.5$
   b) Draw lines AP,BP, CP,DP, EP and FP. These lines are called **secant lines** to the graph $y = x^2 + 1$.
   c) Copy and complete the following table

Point	Coordinates	Line	gradient
P	(0, 1)	-	-
A		AP	

B		BP	
C		CP	
D		DP	
E		EP	
F		FP	

d) As the points get closer to point P, what value does the gradient of the secant lines seem to be approaching?

e) Draw the line that passes through P and that has the gradient you found in part (d). This line is called the **tangent line** to the graph at $y = x^2 + 1$ at P.

2. Consider the graph of $y = x^2$

a) Complete the table of values

Coordinates of P	coordinates of Q	Gradient of PQ
(2, 4)	(4, 16)	
(2, 4)	(3.5, …)	
(2,4)	(3, …)	
(2,4)	(2.5, …)	
$((x, x^2)$	$(x+h), (x+h)^2$	

b) From the last point, what expression do you get for the gradient of PQ? Simplify.

c) Each of the points in your table of values was moving closer and closer to P.

What happens to your expression for gradient if the value of h is too small such that it can be ignored?

3. Repeat the same process in question 2 for the graph of $y = x^3$.

Coordinates of P	coordinates of Q	Gradient of PQ
(2, 8)	(4, 16)	
(2, 8)	(3.5, …)	
(2,8)	(3, …)	
(2,8)	(2.5, …)	
$((x, x^3)$	$(x+h), (x+h)^3$	

4. Hence, write an expression for the gradient of the secant line for the function $y = 3x + 4$

In general, the gradient of a function of $y = x^n$at a point P is given by $\boldsymbol{nx^{n-1}}$, where n is a real number.

$\boldsymbol{nx^{n-1}}$ is also called the derivative of the function $y = x^n$.

If $y = x^n$ then $\frac{dy}{dx} = y = x^n$.

The process of obtaining the derivative of a function is called **differentiation.**

5. Look back at question 1-4 and confirm if this general rule works. Hence use the general rule to find the gradient at P for $y = x^6$ $y = 3x^2$ $y = x^{-2}$.
6. Find the derivative of the following:
   a) $y = 2x^2$
   b) $y = x^6$
   c) $y = 2x^{-3}$
   d) $y = \sqrt{x}$
   e) $y = \frac{1}{x^2}$

Rules of differentiation:

1. From the investigation above, we found that the derivative of $y = 3x + 4$ is 3. This leads us to the constant rule**: the derivative of any constant is zero.**
2. The derivative of $y = x^3 + x^2$ is $3x^2 + 2x$ which leads us to the rule: **The derivative of the sum or difference of two or more terms is equal to the sum or difference of the derivatives of the terms.**
3. $\boldsymbol{y', f'(x) or \frac{dy}{dx}}$ are the various ways of denoting the derivative**.**

Example

Differentiate $f(x) = 3x^2 - 7x$

In this case $h(x) = 3x^2$ and $g(x) = 7x$ and so $\frac{dh}{dx} = 6x$ and $\frac{dg}{dx} = 7$. Therefore, $\frac{df}{dx} = 6x - 7$

Exercise 1

1. Differentiate each function.

   a) $f(x) = 4x^3 + 2x^2 - 3$
   b) $f(x) = 3\sqrt{x} + 8$
   c) $f(x) = (x-2)(x+4)$
   d) $f(x) = \frac{4x^3+2x^2-3}{x}$

2. Find the gradient function of the following:

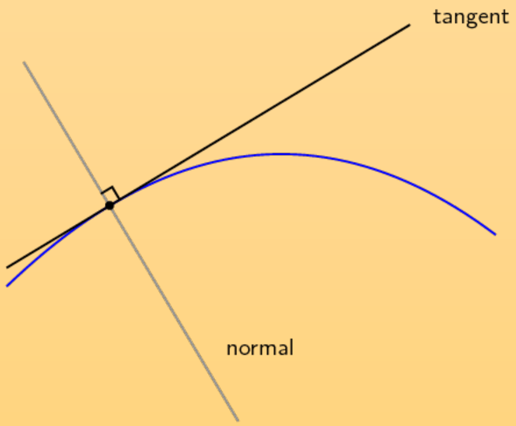
   a) $y = 4x + 2$
   b) $y = x^3 + 2x^2 - 8$
   c) $y = 8$

3. Find $\frac{dy}{dx}$ of the following

   a) $y = x^2(x+2)$
   b) $y = \frac{1}{2}x^2 + \frac{3}{4}x$

Equation of tangents and normal to a curve

A **tangent** is a line that just touches the curve at a point while the **normal** is a line perpendicular to the tangent at that point.



Consider a function $f(x)$ such as that shown in Figure 1. When we calculate the derivative, f', of the function at a point $x = a$ say, we are finding the gradient of the tangent to the graph of that function at that point. Figure 1 shows the tangent drawn at $x = a$. The gradient of this tangent is $f'(a)$.

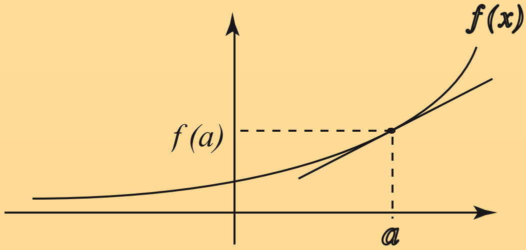


Figure 1. The tangent drawn at $x = a$ has gradient $f'(a)$.

We will use this information to calculate the equation of the tangent to a curve at a particular point, and then the equation of the normal to a curve at a point.

Key point

$f'(a)$ is the gradient of the tangent drawn at $x = a$.

Calculating the equation of a tangent

Example

Suppose we wish to find the equation of the tangent to

$$f(x) = x^3 - 3x^2 + x - 1$$

at the point where $x = 3$.

When $x = 3$ we note that

$$f(3) = 3^3 - 3.3^2 + 3 - 1 = 27 - 27 + 3 - 1 = 2$$

So the point of interest has coordinates (3,2).

The next thing that we need is the gradient of the curve at this point. To find this, we need to differentiate $f(x)$:

$$f'(x) = 3x^2 - 6x + 1$$

We can now calculate the gradient of the curve at the point where $x = 3$.

$$f'(3) = 3.3^2 - 6.3 + 1 = 27 - 18 + 1 = 10$$

So we have the coordinates of the required point, (3,2), and the gradient of the tangent at that point, 10.

What we want to calculate is the equation of the tangent at this point on the curve. The tangent must pass through the point and have gradient 10. The tangent is a straight line and so we use the fact that the equation of a straight line that passes through a point (x_1,y_1) and has gradient m is given by the formula

To determine the equation of a tangent to a curve:

1. Find the derivative using the rules of differentiation.
2. Substitute the x-coordinate of the given point into the derivative to calculate the gradient of the tangent.
3. Substitute the gradient of the tangent and the coordinates of the given point into an appropriate form of the straight line equation.
4. Make y the subject of the formula.

The normal to a curve is the line perpendicular to the tangent to the curve at a given point.

$$m_{tangent} \times m_{normal} = -1$$

Key point

The equation of a straight line that passes through a point (x_1,y_1) and has gradient m is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Suppose we wish to find points on the curve $y(x)$ given by

$$y = x^3 - 6x^2 + x + 3$$

where the tangents are parallel to the line $y = x + 5$.

If the tangents have to be parallel to the line then they must have the same gradient. The standard equation for a straight line is $y = mx + c$, where m is the gradient. So what we gain from looking at this standard equation and comparing it with the straight line $y = x + 5$ is that the gradient, m, is equal to 1. Thus the gradients of the tangents we are trying to find must also have gradient 1.

We know that if we differentiate $y(x)$ we will obtain an expression for the gradients of the tangents to $y(x)$ and we can set this equal to 1.
Differentiating, and setting this equal to 1 we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + 1 = 1$$

from which

$$3x^2 - 12x = 0$$

This is a quadratic equation which we can solve by factorisation.

$$3x^2 - 12x = 0$$
$$3x(x - 4) = 0$$
$$3x = 0 \quad \text{or} \quad x - 4$$
$$x = 0 \quad \text{or} \quad x = 4$$

Now having found these two values of x we can calculate the corresponding y coordinates. We do this from the equation of the curve: $y = x^3 - 6x^2 + x + 3$.

when $x = 0$, $y = 0^3 - 6.0^2 + 0 + 3 = 3$.

when $x = 4$, $y = 4^3 - 6.4^2 + 4 + 3 = 64 - 96 + 4 + 3 = -25$.

So the two points are $(0,3)$ and $(4,-25)$
These are the two points where the gradients of the tangent are equal to 1, and so where the tangents are parallel to the line that we started out with, i.e. $y = x + 5$.

Exercise 2

1. For each of the functions given below determine the equation of the tangent at the points indicated.

a) $f(x) = 3x^2 - 2x + 4$ at $x = 0$ and 3.
b) $f(x) = 5x^3 + 12x^2 - 7x$ at $x = -1$ and 1.
c) $f(x) = xe^x$ at $x = 0$.
d) $f(x) = (x^2 + 1)^3$ at $x = -2$ and 1.
e) $f(x) = sin2x$ at $x = 0$ and $\frac{\pi}{6}$
f) $f(x) = 1 - 2x$ at $x = -3$, 0 and 2.

2. Find the equation of each tangent of the function $f(x) = x^3 - 5x^3 + 5x - 4$ which is parallel to the line $y = 2x + 1$.
3. Find the equation of each tangent of the function $f(x) = x^3+x^2+x+1$ which is perpendicular to the line $2y + x + 5 = 0$.

The equation of a normal to a curve

Investigation: to be done in groups

1. Consider the function $y = x^2 + 1$.
   a) Find its gradient function (its derivative).
   b) What is the gradient of the tangent line at $x = 1$? This is the gradient of the tangent at $x = 1$.
   c) The tangent is passing thorough the point (1, 2), find its equation in terms of $y = mx + c$.
   d) What is the gradient of the normal line at this point?
   e) The normal line is also passing through point (1, 2), find its equation in terms of $y = mx + c$.
2. Use the same steps to find the equation of the tangent line and normal lines for the function $y = x^3 + 2x + 1$ at the point (1, 4)
3. Determine the point on the curve $y = -\frac{1}{2}x^2 + 4$ at which the gradient is 8. Hence, find the equation of the normal to the curve at this point.

In mathematics the word 'normal' has a very specific meaning. It means 'perpendicular' or 'at right angles'.

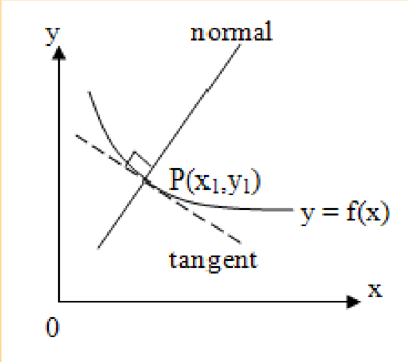


Figure 2. The normal is a line at right angles to the tangent.

If we have a curve such as that shown in Figure 2, we can choose a point and draw in the tangent to the curve at that point. The normal is then at right angles to the curve so it is also at right angles (perpendicular) to the tangent.

We now find the equation of the normal to a curve. There is one further piece of information that we need in order to do this.

Key point

If two lines, with gradients m_1 and m_2 are at right angles then $m_1 m_2 = -1$.

Example

Suppose we wish to find the equation of the tangent and the equation of the normal to the curve

$$y = x + \frac{1}{x}$$

at the point where $x = 2$.

First of all we shall calculate the y coordinate at the point on the curve where $x = 2$:

$$y = 2 + \frac{1}{2} = \frac{5}{2}$$

Next we want the gradient of the curve at the point $x = 2$. We need to find $\frac{dy}{dx}$

Noting that we can write y as $y = x + x^{-1}$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

Furthermore, when $x = 2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{4} = \frac{3}{4}$$

This is the gradient of the tangent to the curve at the point $(2, \frac{5}{2})$. We know that the standard equation for a straight line is

$$\frac{y - y_1}{x - x_1} = m$$

With the given values we have

$$\frac{y - \frac{5}{2}}{x - 2} = \frac{3}{4}$$

Rearranging

$$\begin{aligned} y - \frac{5}{2} &= \frac{3}{4}(x - 2) \\ 4\left(y - \frac{5}{2}\right) &= 3(x - 2) \\ 4y - 10 &= 3x - 6 \\ 4y &= 3x + 4 \end{aligned}$$

So the equation of the tangent to the curve at the point where $x = 2$ is $4y = 3x + 4$.

Now we need to find the equation of the normal to the curve.

Let the gradient of the normal be m_2. Suppose the gradient of the tangent is m_1. Recall that the normal and the tangent are perpendicular and hence $m_1 m_2 = -1$. We know $m_1 = \frac{3}{4}$. So

$$\frac{3}{4} \times m_2 = -1$$

and so

$$m_2 = -\frac{4}{3}$$

So we know the gradient of the normal and we also know the point on the curve through which it passes, $\left(2, \frac{5}{2}\right)$.

As before,

$$\begin{aligned} \frac{y - y_1}{x - x_1} &= m \\ \frac{y - \frac{5}{2}}{x - 2} &= -\frac{4}{3} \end{aligned}$$

Rearranging

$$\begin{aligned} 3\left(y - \frac{5}{2}\right) &= -4(x-2) \\ 3y - \frac{15}{2} &= -4x + 8 \\ 3y + 4x &= 8 + \frac{15}{2} \\ 3y + 4x &= \frac{31}{2} \\ 6y + 8x &= 31 \end{aligned}$$

This is the equation of the normal to the curve at the given point.

Example

Consider the curve $xy = 4$. Suppose we wish to find the equation of the normal at the point $x = 2$. Further, suppose we wish to know where the normal meets the curve again, if it does.

Notice that the equation of the given curve can be written in the alternative form $y = \frac{4}{x}$. A graph of the function $y = \frac{4}{x}$ is shown in Figure 3.

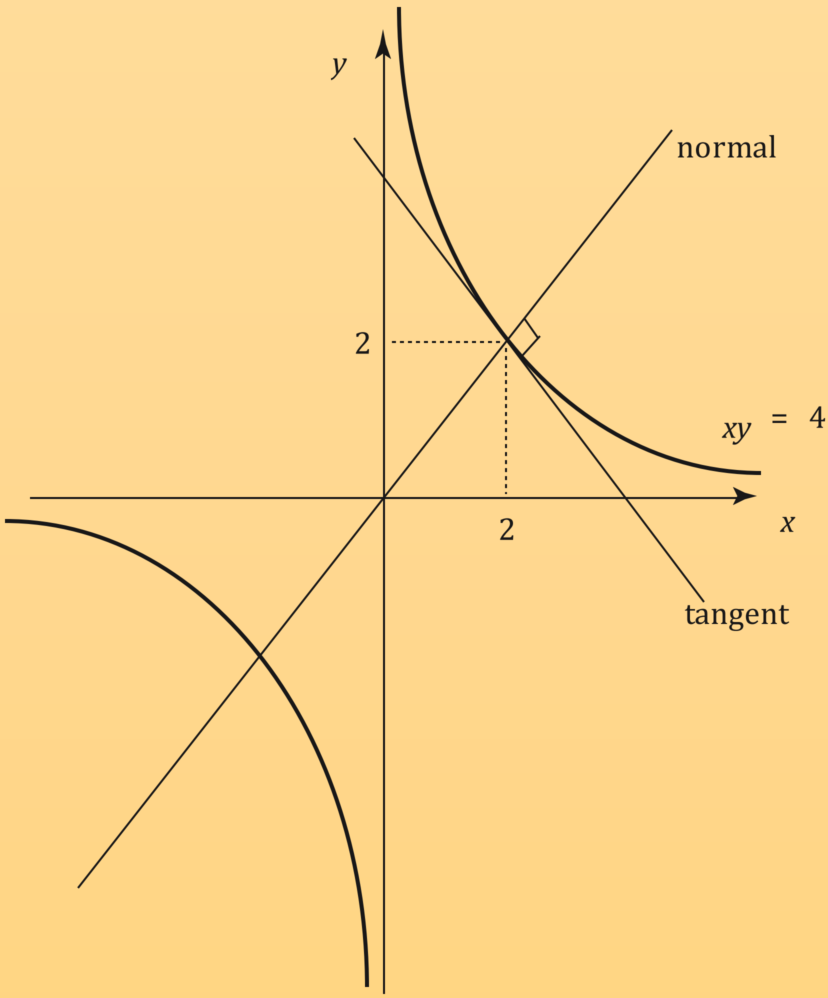


Figure 3. A graph of the curve $xy = 4$ showing the tangent and normal at $x = 2$.

From the graph we can see that the normal to the curve when $x = 2$ does indeed meet the curve again (in the third quadrant). We shall determine the point of intersection. Note that when

$x = 2,\ y = \frac{4}{2} = 2$.

We first determine the gradient of the tangent at the point $x = 2$. Writing

$$\begin{aligned} y &= \frac{4}{x} \\ &= 4x^{-1} \end{aligned}$$

and differentiating, we find

$$\begin{aligned} \frac{dy}{dx} &= -4x^{-2} \\ &= -\frac{4}{x^2} \end{aligned}$$

Now, when $x = 2$ $\frac{dy}{dx} = -\frac{4}{4} = -1.$

So, we have the point (2, 2) and we know the gradient of the tangent there is -1. Remember that the tangent and normal are at right angles and for two lines at right angles the product of their gradients is -1. Therefore we can deduce that the gradient of the normal must be $+1$. So, the normal passes through the point (2, 2) and its gradient is 1.

As before, we use the equation of a straight line in the form:

$$\begin{aligned} \frac{y - y_1}{x - x_1} &= m \\ \frac{y-2}{x-2} &= 1 \\ y - 2 &= x - 2 \\ y &= x \end{aligned}$$

So the equation of the normal is $y = x$.

We can now find where the normal intersects the curve $xy = 4$. At any points of intersection both of the equations

$$xy = 4 \text{ and } y = x$$

are true at the same time, so we solve these equations simultaneously. We can substitute $y = x$ from the equation of the normal into the equation of the curve:

$$\begin{aligned} xy &= 4 \\ x \cdot x &= 4 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

So we have two values of x where the normal intersects the curve. Since $y = x$ the corresponding y values are also 2 and −2. So our two points are (2, 2), (−2, −2). These are the two points where the normal meets the curve. Notice that the first of these is the point we started off with.

Exercise 3

1. For each of the functions given below determine the equations of the tangent and normal at each of the points indicated.

   a) $f(x) = x^2 + 3x + 1$ at $x = 0$ and 4.

   b) $f(x) = 2x^3 - 5x + 4$ at $x = -1$ and 1.

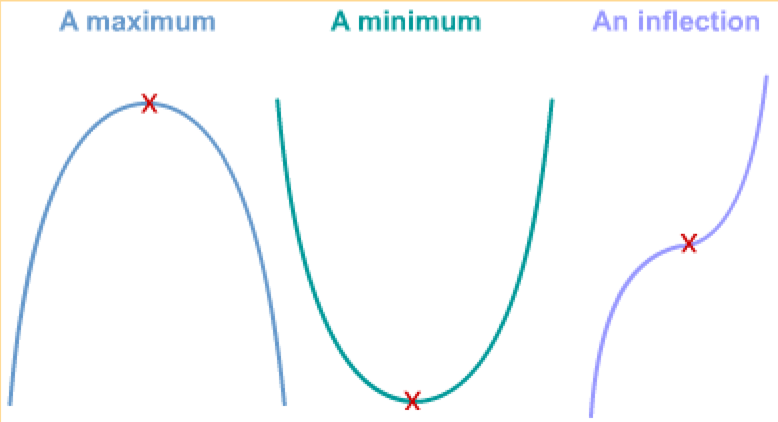
2. Find the equation of each normal of the function $f(x) = \frac{1}{3}x^3 + x^2 + x - \frac{1}{3}$ which is parallel to the line $y = -\frac{1}{4}x + \frac{1}{3}$.

3. Find the x co-ordinate of the point where the normal to $f(x) = x^2 - 3x + 1$ at $x = -1$ intersects the curve again.

Stationary points

The **stationary points** of a function are important in describing how that function works. Finding the stationary points of a function is a useful exercise as you can use them in sketching the function or if you need to locate where the function changes direction. The three types of stationary points are:

- a **maximum**
- a **minimum**
- a **point of inflection**



An important point made in that study guide is that **stationary points are located where the gradient of the function is equal to zero**. As the first derivative of a function describes its gradient, you can use the **first derivative** of the function to find out information about any stationary point the function may have.

To find the location of any stationary points of a function you are looking at, use the following steps:

Step 1: Differentiate the function to find $\frac{dy}{dx}$

Step 2: Set the derivative equal to zero and solve the equation to find value(s) for x. In other words the x-coordinates of any stationary points of your function are the solutions of the equation:

$$\frac{dy}{dx} = 0$$

Step 3: Substitute these values for x back into your **original** function to find the corresponding y-coordinates.

Locating stationary points requires a good working use of differentiation.

Investigation

1. Consider the function $y = x^2 + 2x + 3$,
   a) Find its gradient function.

   Complete the table of values for the values of its gradient

x	-4	-3	-2	-1	0	1	2	3
gradient								

   b) When is the gradient zero? What is the value of the gradient before and after this point?
   c) Draw the graph of the function $y = x^2 + 2x + 3$ on graph paper or a graph plotter. What observation can you make from the shape of the graph and the value of the gradient at various points?

2. Follow the same steps for the function $y = -x^2 + 4x + 2$ for $-1 \leq x \leq 5$.
3. Follow the same steps for $y = x^3 - 4 \;\; for \; -2 \leq x \leq 2$

Note:

1. The point at which the gradient is zero is called a **stationary point**.
2. When the gradient changes from **positive values to zero to negative values**, the stationary point is called a **maximum point.**
3. When the gradient changes from **negative values to zero to positive values**, the stationary point is called a **minimum point.**
4. When the gradient changes from **positive values to zero to positive values**, or **negative values to zero to negative values**, the stationary point is called a **point of inflection.**

This can be summarized below as:

Type of point	**Gradient to the left**	**Gradient at stationary point**	**Gradient to the right**
Maximum	positive	0	Negative
Minimum	Negative	0	Positive
Inflection	Positive	0	Positive
	Negative	0	Negative

Example 1

Identify the stationary points of the function $y = x^3 - 3x + 2$

Solution

$$\frac{dy}{dx} = 3x^2 - 3$$

Gradient of stationary point equals zero, $\frac{dy}{dx} = 0$

Therefore,

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$x = 1 \ or - 1$$

At $x = 1, y = 0$ and at $x = -1, y = 4$

There are 2 stationary points, (1, 0) and (-1,4)

x	0	1	2
$\frac{dy}{dx}$	-3	0	9
sign	negative	0	positive

(1, 0) is a minimum point

x	-2	-1	0
$\frac{dy}{dx}$	9	0	-3

sign	positive	0	negative

(-1,4)**is a maximum point**

Example 2

Find the location of the stationary points of the function $y = x^3 - 3x + 2$.

Step 1: To solve this questions you must first find the derivative of the functions, this tells you about the gradient of the function. Using the power rule of differentiation:

$$\frac{dy}{dx} = 3x^2 - 3$$

Step 2: Set this result, $3x^2 - 3$, equal to zero and solve the quadratic equation for x.

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0$$

$$3(x - 1)(x + 1) = 0$$

Which gives solutions of $x = -1$ and $x = 1$. This means you have two stationary points, one with an x-coordinate of 1 and the other with an x-coordinate of -1.

Step 3: You can find the corresponding y-coordinates by substituting $x = 1$ and $x = -1$ respectively back into the original function $y = x^3 - 3x + 2$:

At $x = 1$, $\quad y = 1^3 - 3 \cdot 1 + 2 = 0$

So the y-coordinate is 0 and this gives a stationary point at (1, 0).

At $x = -1$, $\quad y = (-1)^3 - 3 \cdot (-1) + 2 = 4$

So the y-coordinate is 4 and this gives us a stationary point at (-1, 4).

So the function $y = x^3 - 3x + 2$ has two stationary points at (1, 0) and (-1, 4).

Example 3

To sketch $y = 2 + 3x^2 - x^3$

$$\frac{dy}{dx} = 6x - 3x^2 \ldots\ldots.(1)$$

At stationary points $\frac{dy}{dx} = 0$

This means $6x - 3x^2 = 0$

Factorising gives $3x(2 - x) = 0$

with solutions $x = 0$ or $x = 2$

From (1) $\frac{d^2y}{dx^2} = 6 - 6x$ which is positive when $x = 0$ and negative when $x = 2$.

Substituting the values of x into $y = 2 + 3x^2 - x^3$:

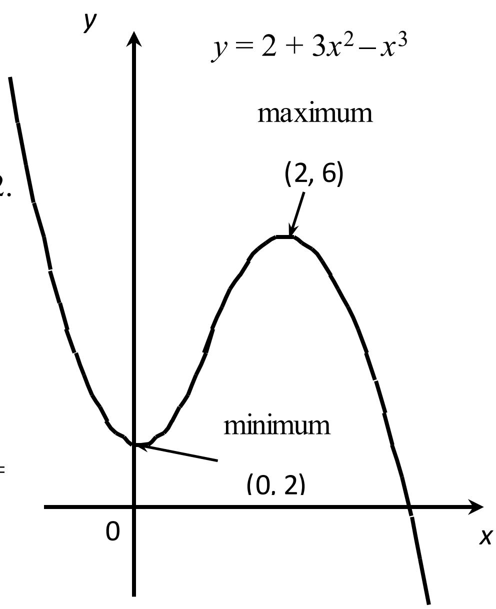
$x = 0$ gives $y = 2$ and $x = 2$ gives $y = 6$

(0, 2) is a **minimum point** and **(2, 6)** is a **maximum point**

When $x = 0$ $y = 2$

When $y = 0$ $2 + 3x^2 - x^3 = 0$

Solving such cubic equations is difficult and not necessary as it is possible to sketch the curve using just the stationary points and the fact that it **crosses the *y* axis** at **(0, 2).**



Exercise 4: To be done in groups.

For the following functions identify the stationary point and determine its' nature. (Use the example above as a discussion point)

1. $y = 2x^3 + 5$
2. $y = 2x^2 - 4x + 3$
3. $y = (3 - 2x)(x + 5)$
4. $y = 2x^3 - 6x$
5. $y = x^5 - 15x^3 + 3$

Exercise 5

For each of the curves whose equations are given below:

- find each stationary point and what type it is;
- find the co-ordinates of the point(s) where the curve meets the x and y axes;
- sketch the curve;
- check by sketching the curve on your graphic calculator.

1	$y = x^2 - 4x$	2	$y = x^2 - 6x + 5$
3	$y = x^2 + 2x - 8$	4	$y = 16 - x^2$
5	$y = 6x - x^2$	6	$y = 1 - x - 2x^2$
7	$y = x^3 - 3x^2$	8	$y = 16 - x^4$
9	$y = x^3 - 3x$	10	$y = x^3 + 1$

For each of the curves whose equations are given below:

- find each stationary point and what type it is;
- find the co-ordinates of the point where the curve meets the y axis;
- sketch the curve;
- check by sketching the curve on your graphic calculator.

11	$y = x^3 + 3x^2 - 9x + 6$	12	$y = 2x^3 - 3x^2 - 12x + 4$
13	$y = x^3 - 3x - 5$	14	$y = 60x + 3x^2 - 4x^3$
15	$y = x^4 - 2x^2 + 3$	16	$y = 3 + 4x - x^4$

Kinematics **(Application of differentiation in the calculation of velocity and acceleration)**

Investigation

1. Consider a car moving from a fixed position O. After t seconds, its displacement in metres is given by $s(t) = 4t + 4$.
   a) Copy and fill in the table of values for this function

t	0	1	2	3
S(t)	4			

   b) On graph paper, draw the graph of the function.
   c) Use your graph or technology to find the velocity of the car at $t = 2$ and $t = 3$(remember velocity of a displacement –time graph is given by gradient)
   d) Find the derivative of the function $s(t) = 4t + 4$
   e) What do you notice?
2. If the displacement function of a particle is $s(t) = 2t^3 + 4t^2 - 8t + 3$
   a) how would we get the velocity of $t = 2$ and $t =$ $without\ drawing\ the\ graph$.
   b) How do we find when the particle is at rest?

From your observations, **the gradient of the displacement function at a certain point gives the velocity at the object at that point**:

$$\boldsymbol{\frac{ds}{dt} = velocity}$$

3. The velocity of a particle in m/s after t seconds is given by $v(t) = 8t + 2$.
   a) Copy and fill in the table of values

t	0	1	2	3
$v(t)$	2			

b) Draw the graph of velocity against time.

c) How do we find the acceleration of the particle at $= 2\ sec, t = 3\ sec$?

d) Find the derivative of $(t) = 8t + 2$. What do can you conclude?

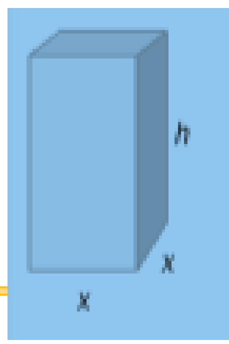
4. Consider the velocity function $v(t) = 5t^2 - t + 3$.
   a) Find the acceleration at $t = 1\ and\ t = 3$.
   b) At what instant is the velocity constant (acceleration is zero)?
5. A ball is thrown vertically upwards. The height of the ball in meters, t seconds after it is released is given by $s(t) = -16t^2 + 40t + 4$ for $t \geq 0\ seconds$.
   a) Find the initial height of the ball.
   b) Show that the height of the ball after 2 seconds is 20 meters.
   c) Find $\frac{ds}{dt}$
   d) Find the initial velocity of the ball in m/s
   e) Find when the velocity of the ball is 0 m/s
   f) Find the maximum height of the ball.

Other application questions

Example

An open box has a square base and surface area of 192 cm^2. Find its dimensions for it to have maximum volume.

Solution



Volume of the box is given by $V = x^2h$

Surface area of the box is $S.A = x^2 + 4xh = 192$

Making h the subject of this expression gives $h = \frac{192 - x^2}{4x}$

Therefore

$$V = x^2\left(\frac{192 - x^2}{4x}\right) = 48x - \frac{1}{4}x^3$$

Maximum volume:

$$\frac{dv}{dx} = 48 - \frac{3}{4}x^2 = 0$$

$$x = \pm 8 \ but\ since\ cannot\ be\ negatgive, \boldsymbol{x} = \mathbf{8}$$

Determine if $x = 8, is\ a\ maximum\ point$

x	7	8	9
$\frac{dv}{dx}$	11.25	0	-12.75

Yes, it is a maximum point

The dimensions are 8cm by 8 cm by 4 cm

Use the example as a discussion point.

A closed cylindrical metal tin has a volume of $250\pi\ cm^3$. If the area of the metal used is to be a minimum, find the radius of the tin.

Integration

Investigation: to be done in groups

1. Consider $f(x) = x^2 + 2x + 3$, and $g(x) = x^2 + 2x$ and $h(x) = x^2 + 2x - 5$
   a) Find the derivative of $f(x), g(x) and\ h(x)$
   b) What do you notice? Why do you think it so?
   c) Give 2 other examples of functions with the same derivative obtained in (a).
2. Given the functions $f(x) = \frac{1}{2}x^2 + 2x + 5$, $g(x) = \frac{1}{2}x^2 + 2x - 10$

$$h(x) = \frac{1}{2}x^2 + 2x,$$

a) Find the derivatives.
b) What do you notice? What pattern can you find to get the function f(x) from the derived function $f'(x)$?

3. Given $f'(x) = 3x^2$, find 2 possible values of $f(x)$

In general, if $\boldsymbol{\frac{dy}{dx} = x^n, then\ y = \frac{x^{n+1}}{n+1} + c, where\ c\ is\ a\ constant\ and\ n \neq -1}$

This process of reversing differentiation is called integration.

Exercise 5

1. Integrate the following.

a) $2x^5$
b) $3x + 4$
c) $3x^2 + 2x + 1$
d) $x^{-\frac{1}{2}}$
e) $x^{\frac{1}{3}}$

Example

1. Find the equation of a curve whose gradient function is $2x + 3$ and which passes through the point (0, 1)

$$\frac{dy}{dx} = 2x + 3$$

Integrating this function gives us: $y = x^2 + 3x + c$

In order to get c, we substitute $x = 0,\ y = 1$ in y.

$$1 = 0 + 0 + c$$

$c = 1$, therefore $\boldsymbol{y = x^2 + 3x + 1}$

2. Find the equation of a curve passing through the point (1, -1) and with a gradient function $y = 3x^2 + 2$

3. The derivative of a function is given by $f'(x) = 4x^5 + 8x$. The graph passes through the point (0, 8). Find $f(x)$.

4. The velocity v m/s of a moving object at a time t seconds is given by

$$v(t) = 3t^2 - 2t$$

when $t = 3$, the displacement, s, of the object is 12 meters. Find an expression for s in terms of t.

(Remember that $v(t) = \frac{ds}{dt}$)

Indefinite integrals

In the exercise above, your integrals all had the constant c. This is called an **indefinite integral**. It is represented by $\int f(x)\ dx$ which means integral of f(x) with respect to x.

If the differential of x^3 is $3x^2$, then

$$\int 3x^2 \, \mathrm{d}x = x^3$$

But $3x^2$ is also the differential of x^3 - 1 and x^3 + 8, *etc.* so that this reversal is not unique - we've 'lost' the *constant*! So in general, $3x^2$ is the differential of $(x^3 + k)$ where k is any constant – this is known as the *'constant of integration'*.

We write this as: $\int 3x^2.\mathrm{d}x = x^3 + k$

(Later on, you'll see that if we're given more information, we can work out the value for k, but for now, we just leave it as it is).

Integration Formula

Since integration is the reverse of differentiation, for any polynomial $y(x) = x^n$, we can simply reverse the differentiation procedure, so that the integral is given by

$$\boxed{\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{(n+1)} + k}$$

(except for $n = -1$)

In words: "Add one to the power, then divide by the new power. Then add k."

Examples

1. $\int x^2\,dx \quad = \quad \frac{x^3}{3} + k$

2. $\int 20x^4\,dx \quad = \quad 4x^5 + k$

Variations on Nomenclature

Because constants don't affect the integration, it is common to bring them in front of the integration sign to make things clearer.

For example: $\int abx^3\,dx \quad = \quad ab\int x^3\,dx \quad = \quad ab\frac{x^4}{4} + k$

or: $\int 5q^2\,dq \quad = \quad 5\int q^2\,dq \quad = \quad \frac{5q^3}{3} + k$

Also, the position of the .dx is usually last in the line, but it can, in principle, be anywhere inside the integral. You may sometimes see the .dx written first (usually in Physics textbooks).

For example: Area $= \int dr\,(r^3 - r^5)$ This is identical to: $\int (r^3 - r^5)\,dr$

Roots follow the same rule

6. $\int \sqrt{x}\,dx \; = \; \int x^{½}\,dx = \frac{x^{3/2}}{(3/2)} \; = \; \frac{2\sqrt{x^3}}{3} + k$

Inverse powers also follow the same rule

7. $\int \frac{1}{x^3}\,dx \quad = \quad \int x^{-3}\,dx = \frac{x^{-2}}{-2} \quad = \quad -\frac{1}{2x^2} + k$

This is true so long as the exponent is *not* -1. $\int \frac{1}{x}$ cannot be calculated using this formula because we get a divide-by-zero error.

Other variables

8. $\int 2m^2\, dm \quad = \quad \frac{2m^3}{3} + k$

9. $\int 5\sqrt{\lambda}\, d\lambda \quad = \quad \frac{10\sqrt{\lambda^3}}{3} + k$

10. $\int \frac{1}{\sqrt{\theta}}\, d\theta \quad = \quad \int \theta^{-½}\, d\theta \quad = \quad \frac{\theta^{1/2}}{1/2} \quad = \quad 2\sqrt{\theta} + k$

Sums of terms

Just as in differentiation, a function can by integrated term-by-term, and we only need one constant of integration.

11. $\int 3x^2 + 7x\, dx \quad = \quad \int 3x^2 + \int 7x \quad =$

12. $\int \sqrt{x} + \frac{1}{x^2} + \frac{5}{3}x^2 + 4x^3\, dx =$

Area under a curve

Investigation

1. Consider the function $y = x$
   a) On a graph paper draw the line $y = x$ and the line $x = 3$
   b) The shape formed by the line $x = 0$ and $x = 3$ is a triangle. Calculate the area of the triangle.
   c) Find $\int x\, dx$
   d) Substitute $x = 0$ *and* $x = 3$ into the integral and subtract the former from the latter. This can be written as $\int_0^3 x\, dx$

   What do you get? What do you notice?

Notice

$\int_0^3 x\, dx$ is called a **definite integral**. 0 and 3 are the **limits** of the integration where 0 is the lower limit and 3 is the upper limit.

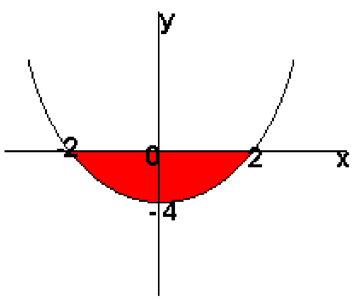
In general, $\int_a^b y\,dx$ is called a **definite integral**. a is the lower limit and b is the upper limit

2. Find the area under the graph of $3x^2 - 4x + 5$ between $x = 2$ and $x = 3$.
3. Find the area of the region bounded by curve $y = x^3 - 3x^2 + 2x$, the x-axis and the lines $x = 1$ and $x = 2$.

We now know how to integrate simple polynomials, but if we want to use this technique to calculate *areas*, we need to know the *limits* of integration. If we specify the limits $x = a \to x = b$, we call the integral a *definite integral.*

To solve a definite integral, we first integrate the function as before (*i.e.* find its indefinite integral), then feed in the 2 values of the limits. Subtracting one from the other gives the *area*.

Example: What is the area between the curve $y = x^2 - 4$ and the x axis?



The shaded area is the area that we want.

We can easily work out that the curve crosses the x axis when x = -2 and x = 2. To find the area, therefore, we integrate the function between -2 and 2.

$$\int_{-2}^{2} (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)$$

$= 16/3 - 16$

$= \underline{-10.67}$ (2d.p.)

Note: the area is negative because it is below the x-axis. Areas above the x-axis, on the other hand, give positive results.

Example

1. What is the area under the curve $y(x) = 2x^2$ between x=1 and x=3?

Area $= \int_{x=1}^{x=3} 2x^2.dx$ we write the limits at the top and bottom of the integration sign

$= \left[\frac{2x^3}{3}\right]_{x=1}^{x=3}$ we use square brackets to indicate we've calculated the indefinite integral

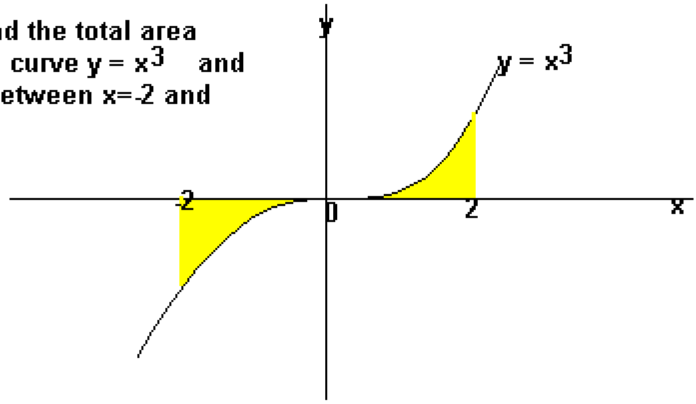
$= 18 - \frac{2}{3}$ feed in the larger value, then the smaller, and subtract the two.

$= 17\frac{1}{3}$ sq. units (compare the approximate value we got graphically of $17\frac{3}{4}$).

Note: when we evaluate a *definite integral* we do not need the constant of integration.

Areas under the x-axis will come out negative and areas above the x-axis will be positive. This means that you have to be careful when finding an area which is partly above and partly below the x-axis.

Example: find the total area between the curve $y = x^3$ and the x-axis between $x=-2$ and $x=2$.



If we simply integrated x^3 between -2 and 2, we would get:

$$\left[\frac{x^4}{4}\right]_{-2}^{2} = 4 - 4 = 0$$

So instead, we have to split the graph up and do two separate integrals.

$$\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 16/4 - 0 = 4$$

$$\int_{-2}^0 x^3\,dx = \left[\frac{x^4}{4}\right]_{-2}^0 = 0 - 16/4 = -4 \qquad \text{(so area is 4).}$$

We then add these two up to get: $\underline{\underline{8 \text{ units}^2}}$

You may also be asked to find the area between the curve and the y-axis. To do this, integrate with respect to y.

Example

What is the enthalpy of a gas at 20 K given that its heat capacity as a function of temperature is given by $C = 2T^2$, over the range $T = 0$ K to 20 K?

You'll learn in chemistry lectures that the enthalpy of a gas, H, is given by the area under the curve of heat capacity *vs* temperature. In most cases, we approximate it by saying that the heat capacity doesn't change much with T, so is in fact a constant. If we take an average value between 0 and 20 K of 10 K, then $C \sim 2\times10^2 = 200$ J K^{-1} mol^{-1}. In this case the enthalpy is just given by

$$H = \int_{T_1}^{T_2} C\mathrm{d}T \quad (\text{with } C = \text{constant} = 200) \quad = \int_{T_1}^{T_2} 200.\mathrm{d}T$$

$$= [200T]_{T_1}^{T_2} \quad = 200(T_2 - T_1) \quad = 200(30 - 0) \quad = \underline{6.0 \text{ kJ mol}^{-1}}$$

However in this question, we are asked for a more accurate answer, and are told C is not constant, it's a function of T.

So $\quad H = \int_{T_1}^{T_2} C\mathrm{d}T$

$$= \int_{T_1=0}^{T_2=20} 2T^2 \ \mathrm{d}T \quad = \left[\frac{2T^3}{3}\right]_0^{20} \quad = \left(\frac{16000}{3}\right) - 0$$

$= \underline{5.3 \text{ kJ mol}^{-1}}$ (compare this with the approximate answer we obtained when we assumed C was constant).

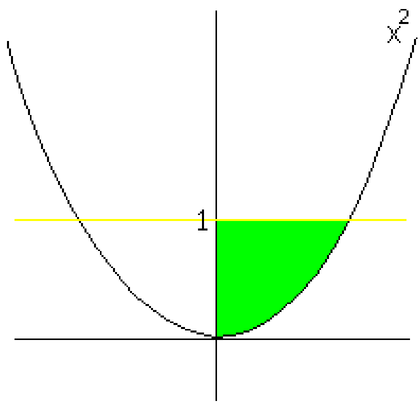
Example

Find the area bounded by the lines $y = 0$, $y = 1$ and $y = x^2$.

$$\text{Area} = \int_0^1 y^{1/2}\,\mathrm{d}y \qquad \text{since } x = y^{1/2}$$

$$= \left[\frac{2y^{3/2}}{3}\right]_0^1$$

$$= 2/3$$



Exercise 6

1. Find the area enclosed by the given curve, the x-axis, and the given ordinates.
   a) The curve $y = x$, from $x = 1$ to $x = 3$.
   b) The curve $y = x^2 + 3$, from $x - 1$ to $x = 3$.
   c) The curve $y = x^2 - 4$, from $x = -2$ to $x = 2$.
   d) The curve $y = x - x^2$ from $x = 0$ to $x = 2$.
2. Find the area contained by the curve $y = x(x-1)(x+1)$ and the x-axis.
3. Calculate the value of

$$\int_{-1}^{1} x(x-1)(x+1)dx$$

   Compare your answer with that obtained in question 2, and explain what has happened.

4. Calculate the value of

$$\int_{0}^{6} (4x - x^2)dx$$

   Explain your answer.