South Sudan

Secondary Additional Mathematics 3

Secondary Additional Mathematics has been written and developed by Ministry of General Education and Instruction, Government of South Sudan in conjunction with Subjects experts. This course book provides a fun and practical approach to the subject of mathematics, and at the same time imparting life long skills to the pupils.

The book comprehensively covers the Secondary 3 syllabus as developed by **Ministry of General Education and Instruction.**

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Secondary Additional Mathematics 3

Student's Book

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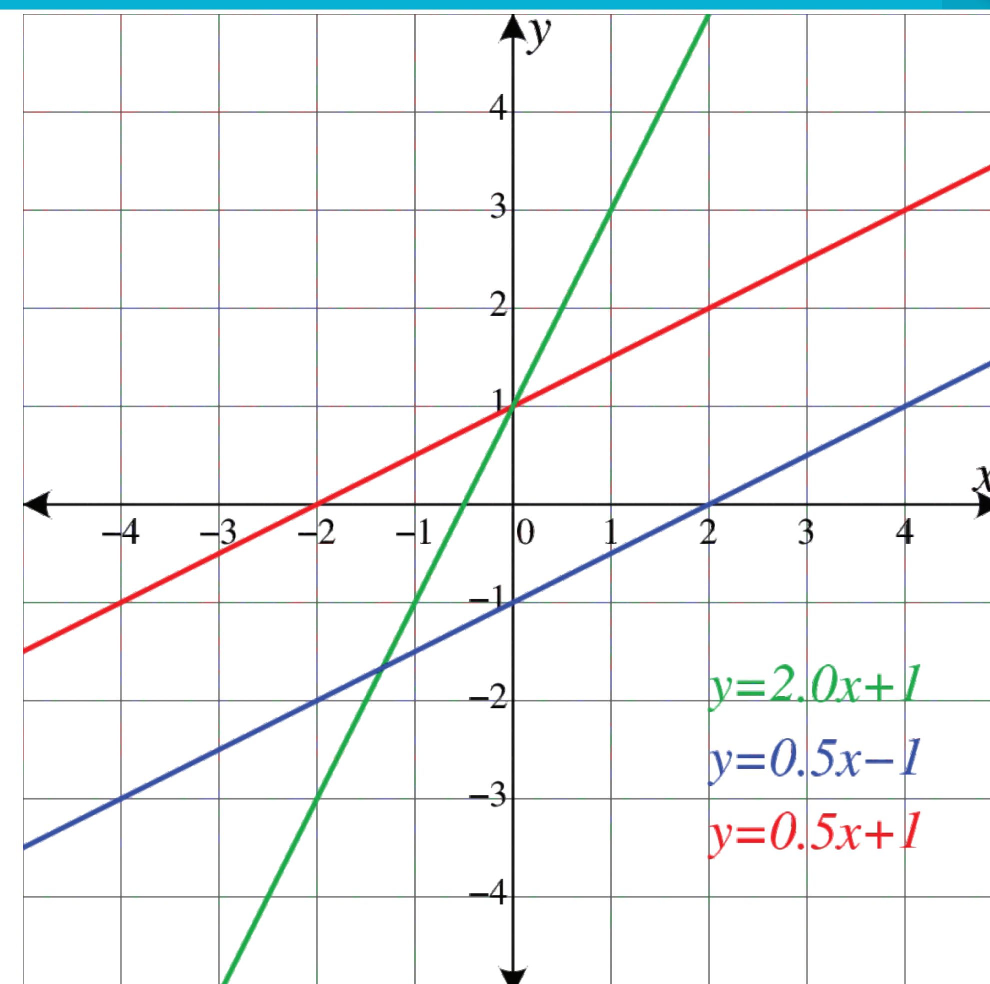

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South Sudan

SECONDARY
3

Additional Mathematics

Student's Book 3

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FOREWORD

I am delighted to present to you this textbook, which is developed by the Ministry of General Education and Instruction based on the new South Sudan National Curriculum. The National Curriculum is a learner-centered curriculum that aims to meet the needs and aspirations of the new nation. In particular, it aims to develop (a) Good citizens; (b) successful lifelong learners; (c) creative, active and productive individuals; and (d) Environmentally responsible members of our society. This textbook, like many others, has been designed to contribute to achievement of these noble aims. It has been revised thoroughly by our Subject Panels, is deemed to be fit for the purpose and has been recommended to me for approval. Therefore, I hereby grant my approval. This textbook shall be used to facilitate learning for learners in all schools of the Republic of South Sudan, except international schools, with effect from 4th February, 2019.

I am deeply grateful to the staff of the Ministry of General Education and Instruction, especially Mr Michael Lopuke Lotyam Longolio, the Undersecretary of the Ministry, the staff of the Curriculum Development Centre, under the supervision of Mr Omot Okony Olok, the Director General for Quality Assurance and Standards, the Subject Panelists, the Curriculum Foundation (UK), under the able leadership of Dr Brian Male, for providing professional guidance throughout the process of the development of National Curriculum and school textbooks for the Republic of South Sudan since 2013. I wish to thank UNICEF South Sudan for managing the project funded by the Global Partnership in Education so well and funding the development of the National Curriculum and the new textbooks. I am equally grateful for the support provided by Mr Tony Calderbank, the former Country Director of the British Council, South Sudan; Sir Richard Arden, Senior Education Advisor of DfID, South Sudan. I thank Longhorn and Mountain Top publishers in Kenya for working closely with the Ministry, the Subject Panels, UNICEF and the Curriculum Foundation UK to write the new textbooks. Finally, I thank the former Ministers of Education, Hon. Joseph Ukel Abango and Hon. Dr John Gai Nyuot Yoh, for supporting me, in my previous role as the Undersecretary of the Ministry, to lead the Technical Committee to develop and complete the consultations on the new National Curriculum Framework by 29 November 2013.

The Ministry of General Education and Instruction, Republic of South Sudan, is most grateful to all these key stakeholders for their overwhelming support to the design and development of this historic South Sudan National Curriculum. This historic reform in South Sudan's education system is intended to benefit the people of South Sudan, especially the children and youth and the future generations. It shall enhance the quality of education in the country to promote peace, justice, liberty and prosperity for all. I urge all Teachers to put this textbook to good use.

May God bless South Sudan. May He help our Teachers to inspire, educate and transform the lives of all the children and youth of South Sudan.

Deng Deng Hoc Yai, (Hon.)
Minister of General Education and Instruction, Republic of South Sudan

Table of Contents

FUNCTIONS

Introduction

Previously, we discussed the relationship between two variables using simple equations such as $y = 2x + 5$. In this equation, the variable y is a function of x. which is written as $y = f(x)$, since $y = 2x + 5$ the equation is written as, $y = f(x) = 2x + 5$ or $f(x) = 2x + 5$

In these functions, the inputs and outputs are real numbers. For instance:

Input x	-1	0	1	2	3	4
Output $f(x) = 2x + 5$	3	5	7	9	11	13

Table 1.1

In this chapter, we will consider broader functions whose input and/or output values may be integers, sets, strings, characters and the like. Suppose A and B are two sets of numbers and a certain rule assigns each element of set A to exactly one element of B. Such a rule is a function.

A function is a rule that assigns each value of a set A to exactly one value of a set B.

The set A, which is the input value, is called a **domain** or object values of the function. The set B, which is the output value, is called the **co-domain or** the range or image values of the function.

Domain and Range of Functions

Domain is the set of all real numbers ($x \in \mathbb{R}$). It consists of all the values of x for which $f(x)$ is well defined. It can also be called the **object.**

The **co-domain** of a function is the set of all real numbers $y = f(x)$ that is well defined by the domain. It is also called the **range** or **image**.

Graphing a function

The relationship between two variables defines the nature of the graph of their relationships. The most common relationships that exist are either linear or non-linear. Non-linear relationships include quadratic, cubic, polynomial, trigonometric and exponential relationships.

As each value of the domain maps to a single value in the range, there will be a unique y-value for each value of x.

By definition, a graph of a function is a set of all points whose coordinates (x, y) satisfy the function $y = f(x)$. In the graph, for each x value there is a corresponding y value obtained by substituting into the expression $f(x)$.

Linear graphs

These are graphs of the functions whose general format is $f(x) = ax + b$ where a and b are constants. They are straight line graphs.

Task 1

i. Draw a graph of the function $f(x) = 2x + 3,\ x \in \mathbb{R}$

Solution

This graph is drawn by obtaining any three values of x (the domain) then solving for corresponding y values to determine some points the graph passes through.

$$f(x) = 2x + 3$$

$f(1) = 2 \times 1 + 3 = 5$ P_1 (1, 5)

$f(0) = 2 \times 0 + 3 = 3$ P_2 (0, 3) (This point is the y-intercept.)

$f(3) = 2 \times 3 + 3 = 9$ P_3 (3, 9)

i. The graph is shown in figure 1.9 below

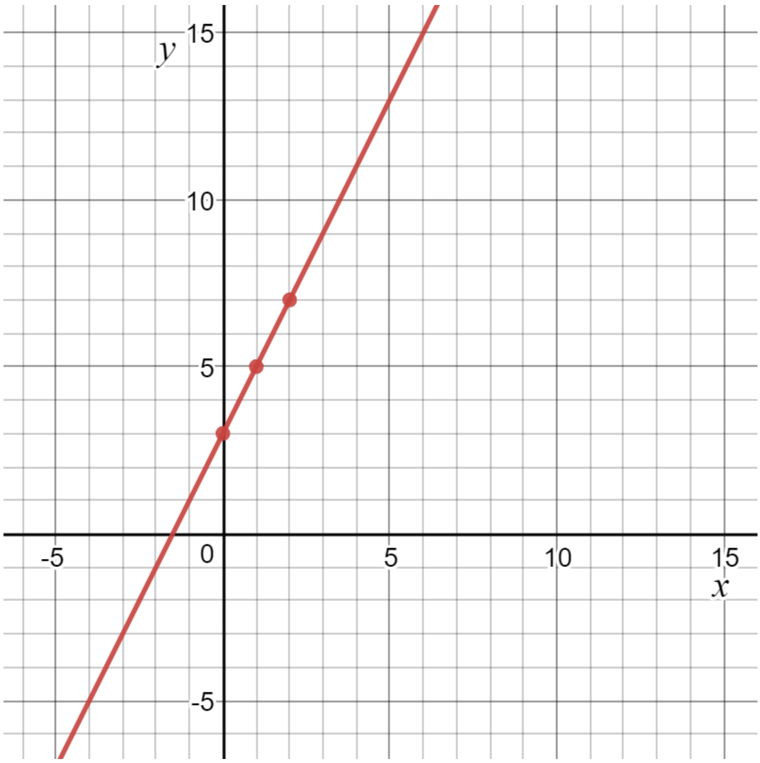


Figure 1.9

Quadratic Graphs

Example

Draw the graph of $y = x^2$, $x \in \mathbb{R}$ and state the range.

Solution

Determine a set of ordered pairs of x (domain) and y (co-domain).

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

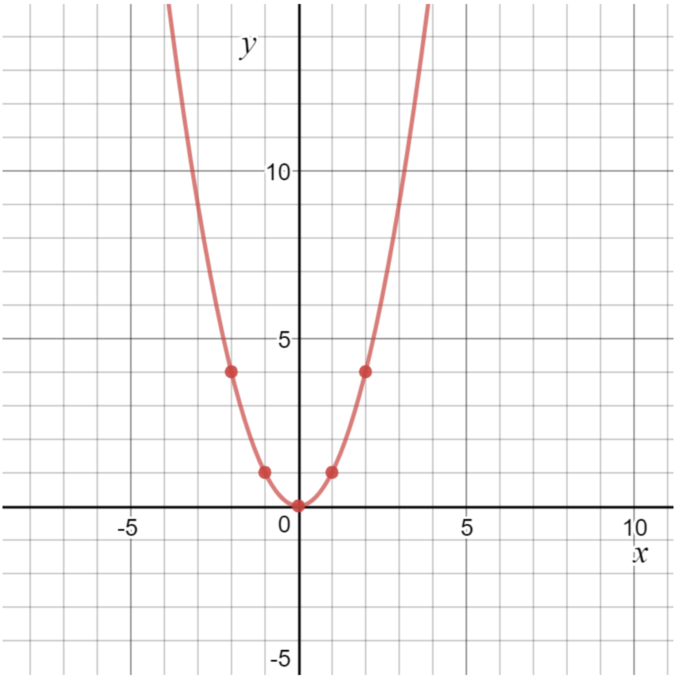


Figure 1.10 **Quadratic graph**

The range of $y = x^2$ is positive reals including zero $\mathbb{R}_0^+$

Graphs with undefined points

Example 4

Draw the graph of $y = 1 + \frac{1}{x}$, $x \in \mathbb{R} \setminus 0$ and state the range.

Solution

$$y = 1 + \frac{1}{x}$$

$$y = \frac{x+1}{x}$$

Note: When $x = 0$ a number is divided by 0 and the result is not defined. Hence $x = 0$ is not in the domain of $y = 1 + \frac{1}{x}$ and y does not exist. The notation $\mathbb{R}\backslash 0$ means all the real numbers excluding zero.

The graph cannot be drawn when $x = 0$. This line $x = 0$ (y-axis) is an asymptote of the curve. Similarly, as the magnitude of x gets very large $\frac{1}{x}$ approaches zero so y approaches 1. There are no values of x for which $y = 1$, so this is another asymptote.

A table of ordered pairs of values for $y = 1 + \frac{1}{x}$ is:

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	-	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$

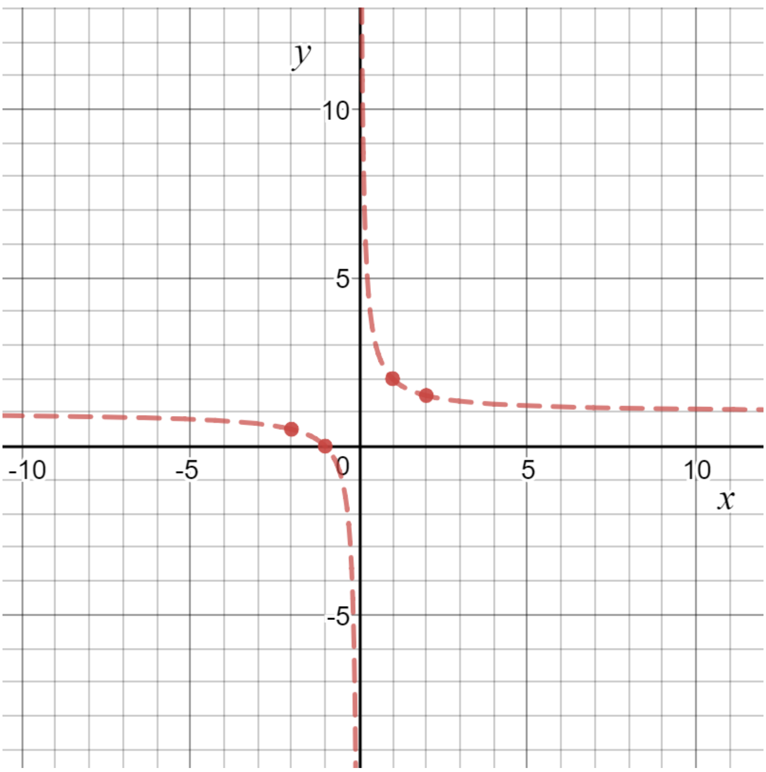


Figure 1.11

The asymptotes are drawn with a dotted line.

The range of $y = 1 + \frac{1}{x}$ is all the real numbers excluding one ($\mathbb{R} \setminus 1$).

Example 5

Draw the graph of $y = \sqrt{x}$, and state the range.

Solution

x	0	1	2	3	4	5	6
y	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$

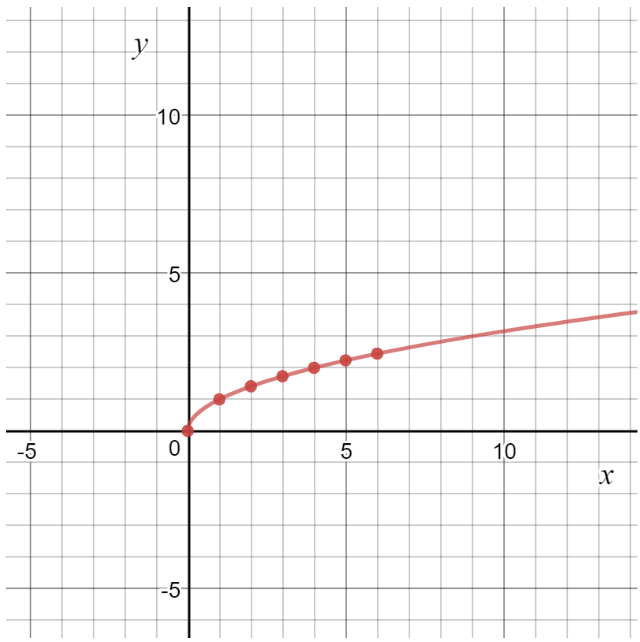


Figure 1.12

This graph is a half parabola.

The range of $y = \sqrt{x}$ is positive reals including zero ($\mathbb{R}_0^+$)

Example 6

You are given the function $y = \sqrt{-3x + 1}$,

i. Determine the values of x for which the function is defined.
ii. Develop a suitable table for drawing the graph of this function
iii. Draw the graph of $y = \sqrt{-3x+1}$

Solution

i. The function is not defined for real numbers with $-3x+1 < 0$ since the square root of a negative number is not a real number.

$$-3x < -1$$

$$x > \frac{1}{3}$$

It is hence defined for all real values of $x \le \frac{1}{3}$.

ii. A possible table of values

x	$\frac{1}{3}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	-1	-2	-3	-4	-5
$y = \sqrt{-3x+1}$	0	$\frac{1}{2}$	1	$\sqrt{\frac{5}{2}}$	2	$\sqrt{\frac{7}{2}}$	$\sqrt{\frac{10}{2}}$	$\sqrt{\frac{13}{2}}$	4

iii. The graph is

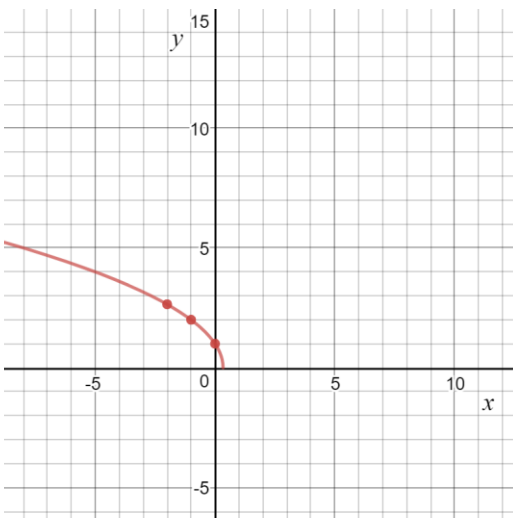


Figure 1.13

Exponential Graphs

Exponential graphs are graphs of the general formula $y = a^x$ where a is a constant and x is an integer.

Example

For the function $y = 3^x$

i. What is the y-intercept?

ii. Draw the graph of $y = 3^x$, $x \in \mathbb{R}$ and state the range.

Solution

i. $y = a^0 = 1$

All exponential graphs of general formula $y = a^x$ pass through (0, 1).

x	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{3^3}$	$\frac{1}{3^2}$	$\frac{1}{3}$	1	3	9	27	81

As x decreases y approaches zero but can never take the value zero so the $x - axis$ is an asymptote of curve $y = 3^x$, and the range is all positive reals (□ ⁺).

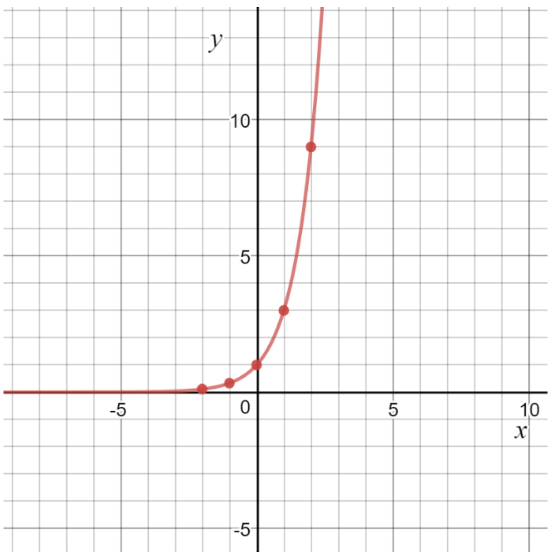


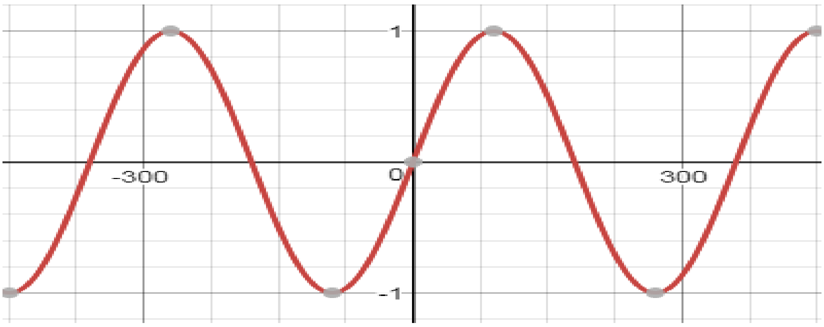
Figure 1.14

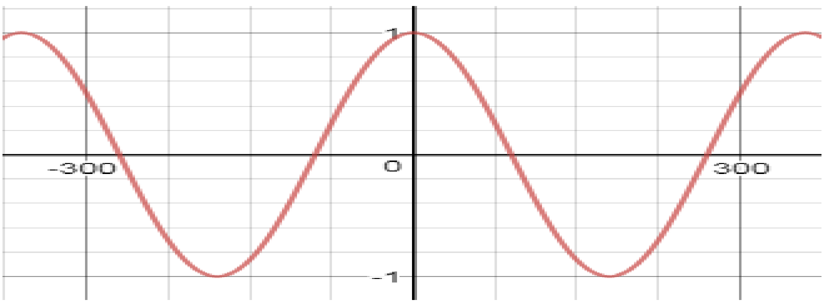
Trigonometric graphs

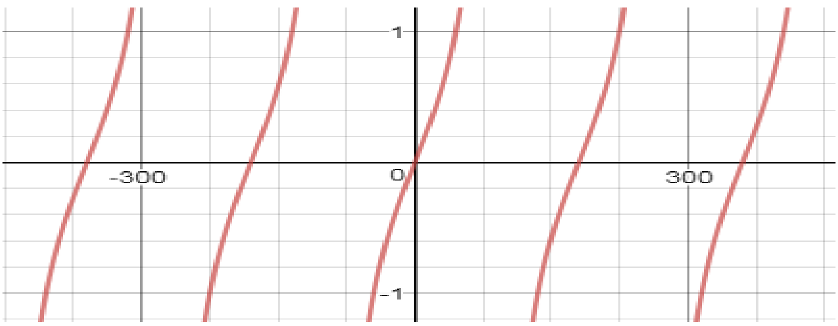
Task

The graphs below represent the trigonometric functions. In groups, match the functions for sine, cosine and tangent with their graphs. Add labels for axes and asymptotes.

State the domain and range for each function.







Further reading activity

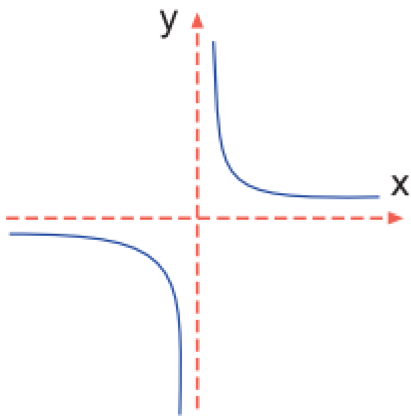
In groups investigate how to draw graphs using software.

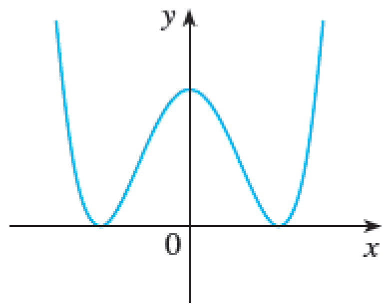
Sketching graphs

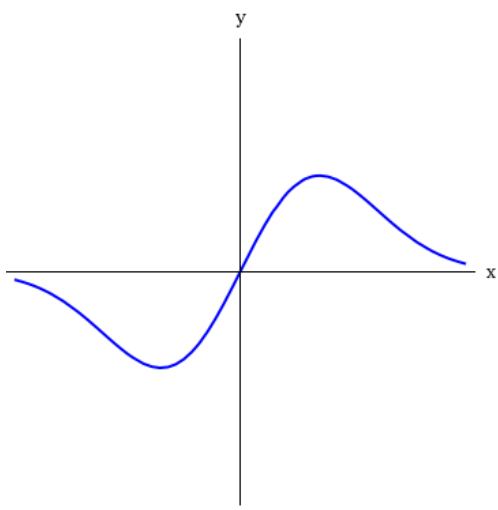
It is often more convenient to draw graphs with technology rather than plotting points by hand. Rather than printing the graphs, it is often more convenient to sketch the graph. A sketch includes the key features:

- The shape of the graph
- Labelled axes
- Any intercepts (i.e. when x and y are zero)
- Any asymptotes (as a dotted line)

Example







Exercise 1.1 (work in pairs and use technology)

1. Find the value of ,
   i. $f(1)$
   ii. $f(3)$
   iii. $f(5)$

for each of the following functions.

a. $f(x) = x^2 - 4x + 3$
b. $f(x) = \sqrt{x}$
c. $f(x) = \sqrt{x^2 + 3x + 1}$
d. $f(x) = x^2\sqrt{2x + 1}$
e. $f(x) = 10 - 3x + 4x^3 + 3x^2$
f. $f(x) = \frac{2}{x^2+2x}$

2. For each of the following functions, determine the domain and range.
   a) $f(x) = \frac{1}{x^2}$
   b) $f(x) = \sqrt{x^2 + 1}$
   c) $f(x) = x\sqrt{x - 2}$
   d) $f(x) = \frac{1}{x^2 - 3}$
   e) $f(x) = \frac{x}{x-2}$
   f) $f(x) = \frac{2}{x^2+6x+9}$
   g) $f(x) = \frac{x^2-16}{x^2+8x+16}$

h) $f(x) = \sqrt{x^2 - 3x - 4}$

i) $f(x) = \frac{3}{x-2}$

3. Draw the graph of the following functions $x \in \mathbb{R}$ and state the range
   a) $y = 4x + 2$
   b) $f(x) = 3x + 2$
   c) $f(x) = 2x - 4$
   d) $f(x) = 7x - 8$
   e) $f(x) = -7 - 4x$
   f) $2x - 2y + 3 = 0$
   g) $4x - 3x + 2y = 0$

   What do you notice?

4. Draw the graph of the following functions $x \in \mathbb{R}$ and state the range
   a) $y = 2x^2 + 3$
   b) $f(x) = x^2 + 4x$
   c) $f(x) = -x^2 + 2x$
   d) $y = 2x^2 - 3x^2 + 4$
   e) $4x^2 + y + 4 = 0$

   What do you notice?

5. Draw the graph of the following functions $x \in \mathbb{R}$ and state the range
   a) $y = x^3 + 3x$
   b) $3x^3 + 2x + y = 0$
   c) $f(x) = x^3 - x^2$

6. Draw the graph of the following functions, state the domain and range.

a) $f(x) = \frac{x^2+x}{x+1}$

b) $y = \frac{x^2+2x}{3x-4}$

c) $\frac{f(x)}{x+1} = 7$

d) $2f(x) + 7x + 3 = 0$

7. Draw the graph of the following functions $x \in \mathbb{R}$ and state the range
   a) $y = 2^x$
   b) $f(x) = 4^{2x}$
   c) $f(x) = 5^{-x}$

8. The electric power P (in watts) delivered by a battery as a function of the resistance R in *Ohms* is $P = \frac{10\ 000R}{(10+R)^2}$.

   Plot the function of power against resistance.

When is a graph not a graph of a function?

When plotting a function each element in the domain takes a unique value in the co-domain/range. Several elements in the domain may result in a single element in the co-domain.

Task

For the graphs (1-7) below, decide whether or not *y* is a function of *x*.

1. The straight line $y = -x + 1$

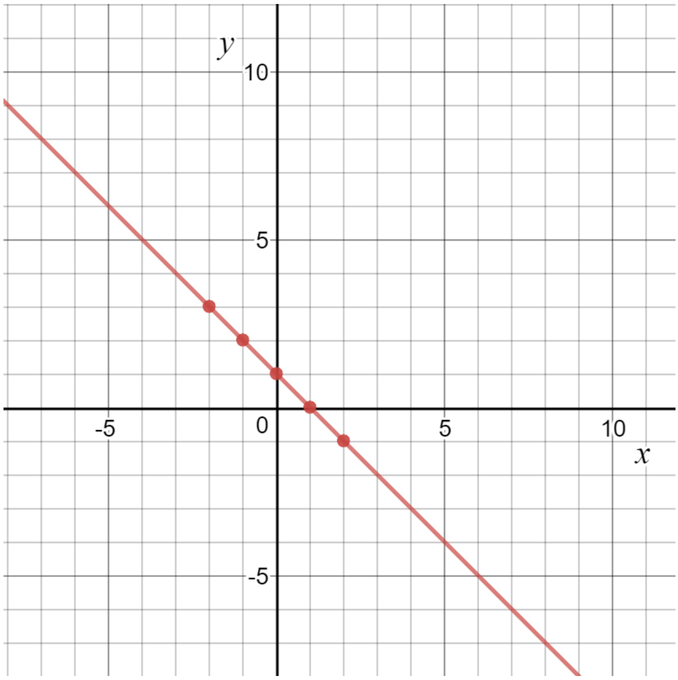


Figure 1.15

2. The quadratic $y = x^2 - 2x - 3$

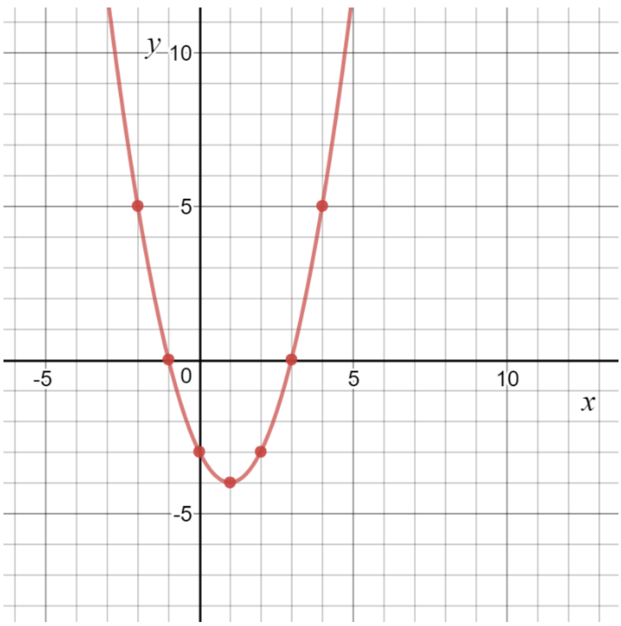


Figure 1.16

3. The graph of the equation $y = \cos x$

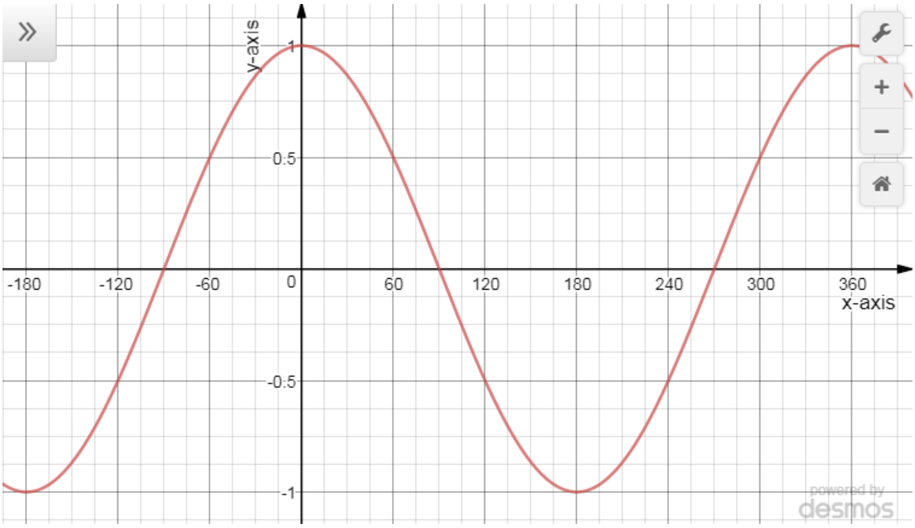


Figure 1.17

4. The e llipse , $\frac{x^2}{25} + \frac{y^2}{9} = 1$

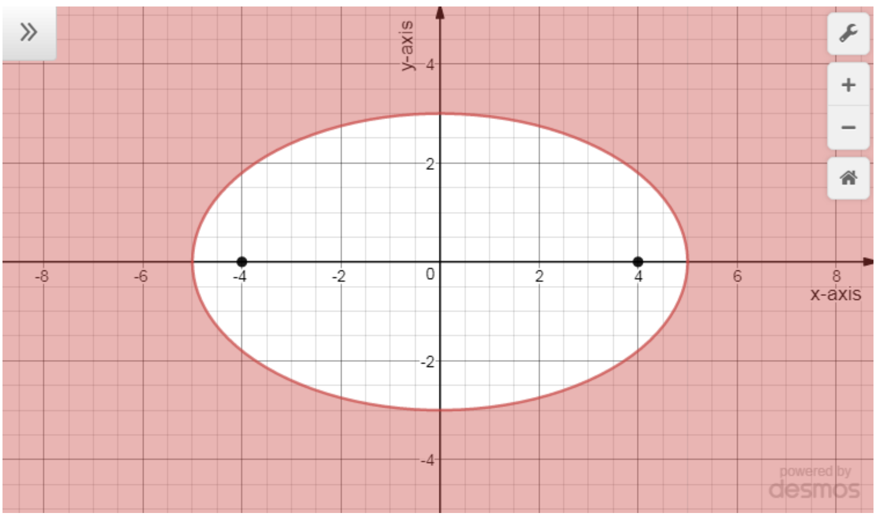


Figure 1.18

5. The h yperbola graph $x^2 - y^2 = 1$

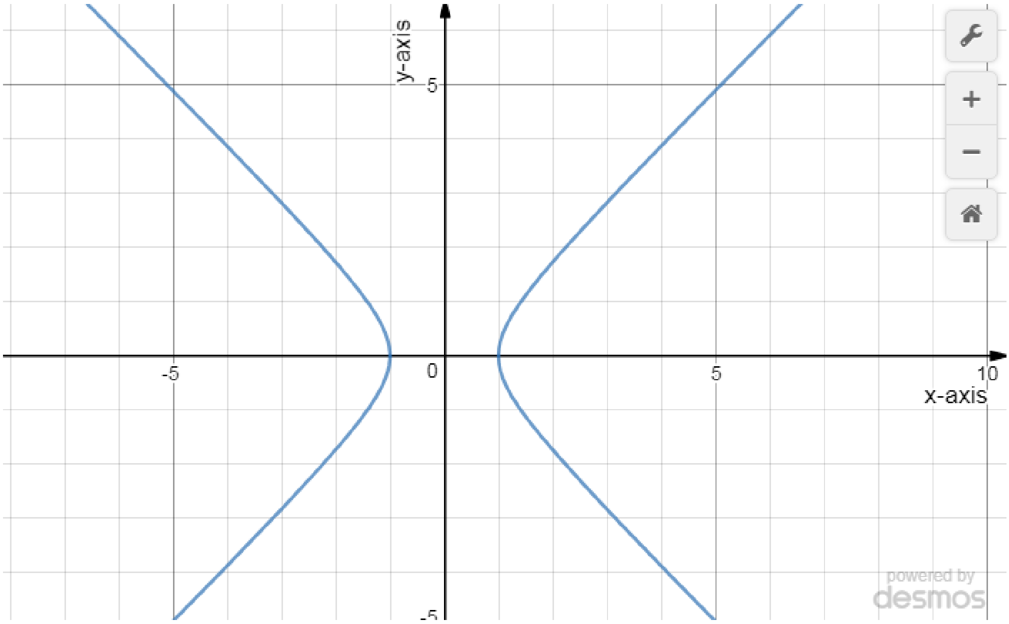


Figure 1.19

6. The circle $x^2 + y^2 = 25$

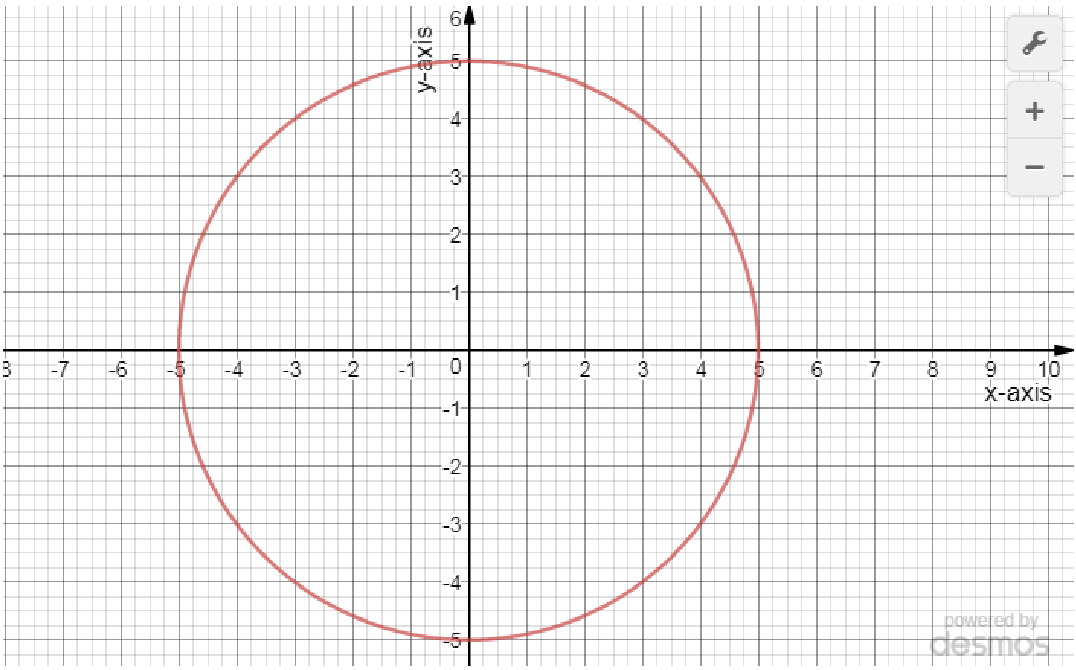


Figure 1.20

7. The graph of equation $x = y^2 - 3$

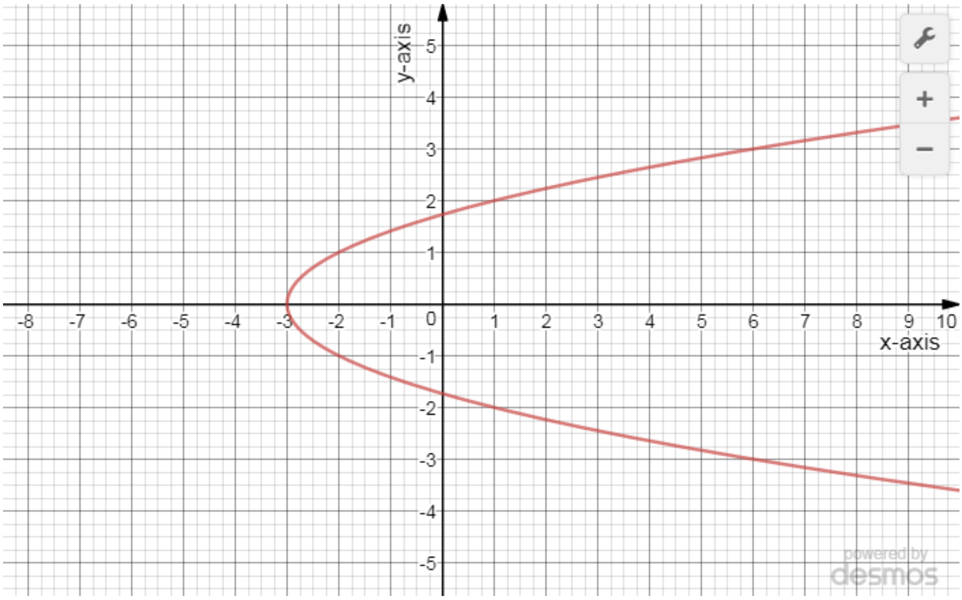


Figure 1.21

Example 1

Without using the graph , determine whether or not the equation of a circle $x^2 + y^2 = 1$ is a function .

Solution

Substituting $x = 0$ in the equation $x^2 + y^2 = 1$,

$$x^2 + y^2 = 1$$

$$0 + y^2 = 1$$

$$y = \pm\sqrt{1} \qquad y = +1 \; or \; y = -1$$

Since there are two values of y when x $= 0$, $x^2 + y^2 = 1$ is not a function.

Exercise 1.2

Determine whether or not the graphs of the following equations are of a function.

1. $y = x^3 + x^2 + 3x + 4$
2. $y = x^3 - x$
3. $y = x^{\frac{1}{2}}$
4. $y^2 = x^2 + 3x + 4$
5. $2x^2 + 2y^2 - 4x + 2y = 0$
6. $3x^2 - 2y^2 + 3xy = 0$
7. $y = 2^x$
8. $x = (y - 3)^2 + 5$

Implicit and Explicit Functions

The word implicit means, a suggestion through indirect expression. An **implicit function** is the one in which a function is defined by a relationship between the variables e.g. $2x + y = 7$.

An **explicit function** is one that is defined clearly and directly e.g. $y = -2x + 7$.

Example 1

The relationship between x and y is defined implicitly as

$x^2 + xy - 2 = 0.$

a) Express this as an explicit function.
b) Determine the domain and range of the function.

Solution

a) $x^2 + xy - 2 = 0$

$xy = 2 - x^2$

$y = \dfrac{2 - x^2}{x}$

$y = \dfrac{2}{x} - x$

b) Recall that $x \div 0$ is undefined. Thus 0 is not part of the domain.

The domain is any real number except 0 which can be written as $\mathbb{R} \backslash 1 = \{(-\infty, 0) \cup (0, \infty)\}$.

As x gets very large, y approaches $-x$, so $y = -x$ is also an asymptote.

The range is any real number .

Example 2

$x^2 + y^2 = 9$ is an equation of a circle.

a) Is it an implicit or explicit equation?
b) Does it represent a function?
c) Which two functions $h(x)$ and $g(x)$ form the equation $x^2 + y^2 = 9$?

Solution

a) It is an implicit equation as there is no direct expression of $f(x)$.
b) Check whether there are unique y values for each value in the domain.

If $x = 0$

$x^2 + y^2 = 9$

$y^2 = 9$

$y = \pm\sqrt{9} = \pm 3, \quad y = -3 \quad and\ y = 3$

As there are two distinct y values the equation is not a function.

c) Rearranging to find an expression for y

$$x^2 + y^2 = 9$$

$$y^2 = 9 - x^2$$

$$y^2 = \sqrt{9 - x^2}$$

$$y = \pm\sqrt{9 - x^2}$$

The two explicit functions are:

$g(x) = -\sqrt{9 - x^2}$ and,

$h(x) = +\sqrt{9 - x^2}$

Exercise 1.3

Where possible, express the following implicit equations as explicit equations.

1. $y = 2x + 2 + 4y$
2. $f(x) = x^2 + 3 + 6$
3. $x^2 + y^2 = 6$
4. $x^3 + y^2 = 5$
5. $7x^6 + 4x^4 + 3y^2 = x$
6. $9xy + 3x^2y + 3x^3y + 3x^2 + 4x + 20 = 0$
7. $xy^2 - 4y = x^2 + 1$
8. $y^2x - 2y = xy$

Using implicit equations to describe functions

It should be clear that not all implicit equations can be expressed as explicit. For instance, the equation $y^2 + 3y + 2x = 0$ cannot be expressed explicitly. This is because one cannot express y as a function of x.

In determining whether or not implicit equations are of a function, one needs to determine the result of the equation at any value x and find out whether a domain-range relationship exists. Technology can also be used to draw the graph of the equation and look to see if there is a unique y value for each value of x.

Example

Determine whether or not $y^3 + 3y + 2x = 0$ is a function.

Solution

Try $x = 0$

$y^3 + 3y + 2x = 0$

$y^3 + 3y = 0$

$y\,(y^2 + 3) = 0$

$$y = 0, \quad or \quad y^2 = -3, \quad y = \pm\sqrt{-3},$$

y is not defined at $x = 0$, since $\sqrt{-3}$ is not a real number.

In this case, 0 does not have a mapping and hence, $y^3 + 3y + 2y = 0$ is not a function.

Exercise 1.4

1. Determine whether or not the following implicit equations are a function.
   a) $y^3 - 3y + 2x = 0$
   b) $4x^4 + 20x^3 + 160x^2 + 80x + 20y = 0$
   c) $yx^2 + 16y = x^2 - 16 - 8xy$
   d) $25\,(x - 3)^2 + 10(y + 2)^2 = 100$
2. Determine the range and the domain for the functions above.

Modulus of a function

The word modulus means the absolute value or magnitude of a number. Modulus of x is written as $|x|$.

In inequalities, if $|x| \leq k$, then $-k \leq x \leq k$.

Example

Draw the graphs of

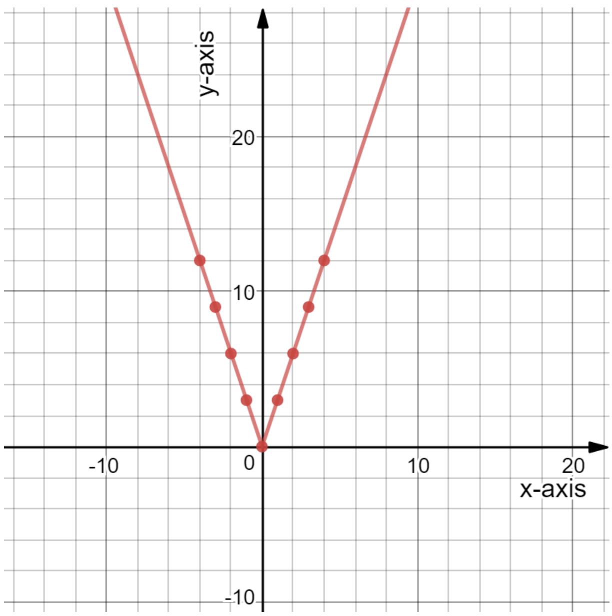
a) $y = f(x) = |3x|$

b) $y = f(x) = |x^2 + 3x|$

Solution

a) $y = f(x) = |3x|$

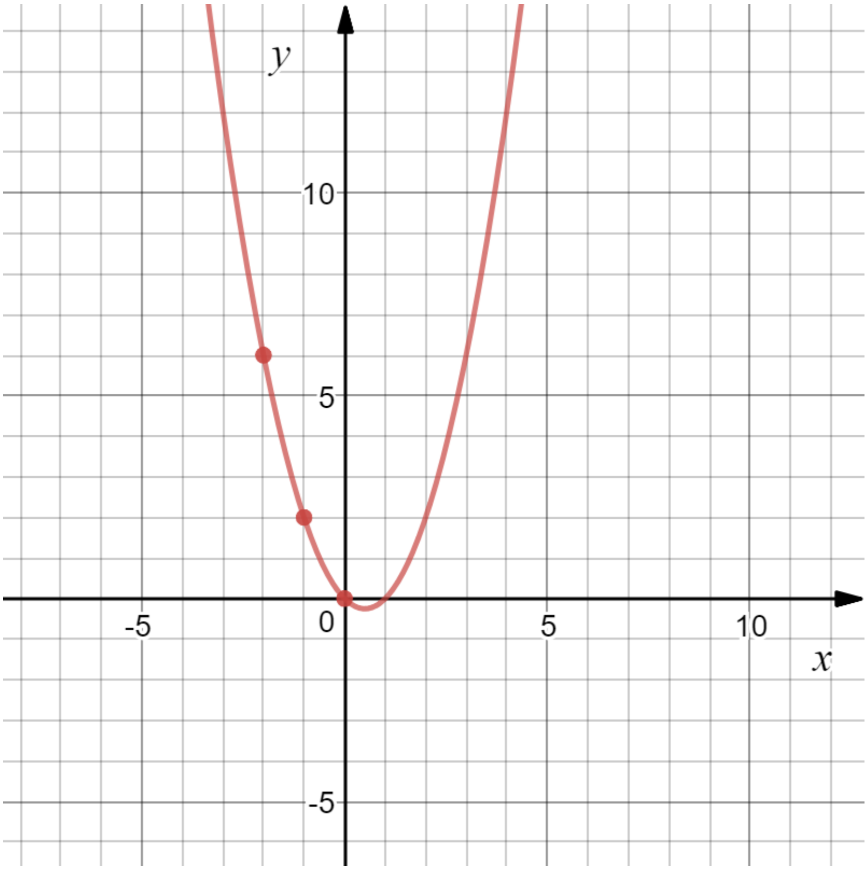
x	-4	-3	-2	-1	0	1	2	3	4
3x	-12	-9	-6	-3	0	3	6	9	12
|3x|	12	9	6	3	0	3	6	9	12



a) $y = f(x) = |x^2 + 3x|$

x	-4	-3	-2	-1	0	1	2
$x^2 - x$	4	0	-2	-2	0	4	10

$\lvert f(x)\rvert$	4	0	2	2	0	4	10



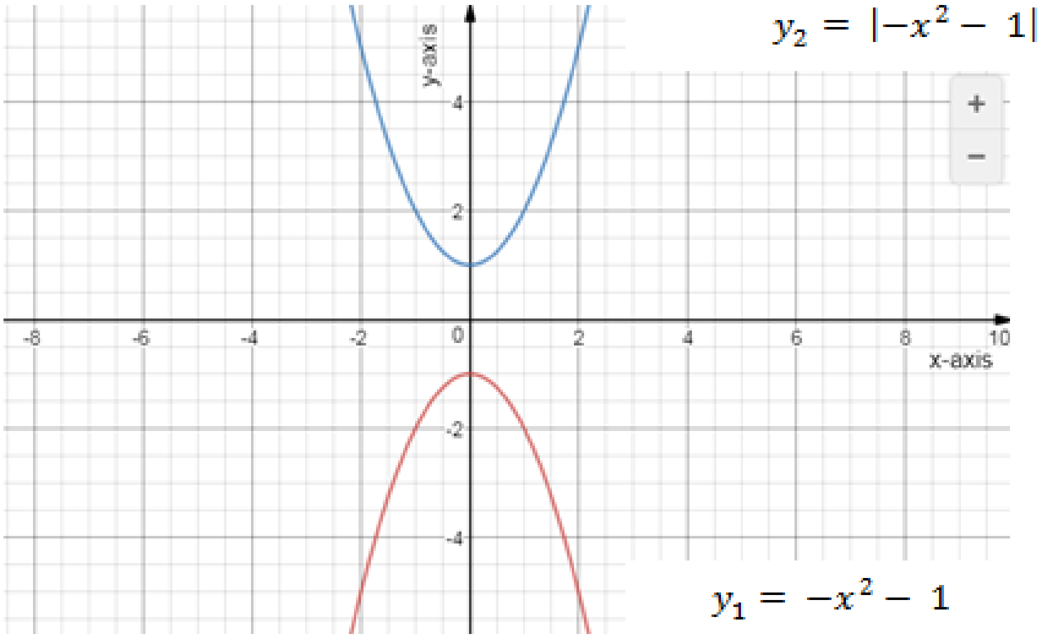
If a function $y = f(x)$ takes both positive and negative values, function $y = |f(x)|$ takes only positive ranges.

Example

i. Draw the graphs of the following functions on the same Cartesian plane.
   a. $y_1 = -x^2 - 1$
   b. $y_2 = |-x^2 - 1|$

ii. What is the relationship between the graphs of $y_1 = f(x)$ and $y_2 = |f(x)|$.

Solution



The negative range in $y_1 = f(x)$ is reflected in the x-axis to obtain a modulus function $y_2 = |y_1| = |f(x)|$.

Exercise 1.5: Work in pairs, use graph plotting software

1. Draw the graphs of the following
   a) $y = x^2$
   b) $y = x^2 - 3x$
   c) $y = f(x)$ for $f(x) = -x^2 + 3x$
   d) $y = |f(x)|$ for $f(x) = -x^2 + 3x$
   e) $y = f(|x|)$ for $f(x) = -x^2 + 3x$
   f) $y = x^3 - 2x^2 - 3x + 2$
   g) $y = |f(x)|$ for $f(x) = x^3 - 2x^2 - 3x + 2$
   h) $y = f(|x|)$ for $f(x) = x^3 - 2x^2 - 3x + 2$
   i) $y^2 = |x|$
   j) $y = \sqrt{|3x|}$
   k) $y = |x| + 2$

l) $y = -|x+3|$

m) $y = |f(x)|$ for $f(x) = x^2 + 3x - 4$

2. Determine the domain and range for each of the following functions.
   a) $f(x) = |x^2 - 3x|$
   b) $f(|x|) = 2x + x$
   c) $f(x) = |\frac{1}{x}|$
   d) $f(x) = \frac{2}{\sqrt{|2-x|}}$
   e) $f(x) = \frac{3x}{\sqrt{|5-3x|}}$

Application of Modulus

The modulus of a function where $y = |f(x)|$ always gives a positive value solution. Such solutions are used to determine countable valuables such as the age of a person or baby; and countable or measurable quantities such as mass. If the law in a relationship ought to give absolute value and the same calculations such a relationship may be expressed as an absolute function.

Some quantities such as temperature have two units that are closely related. One of the quantities provides a negative scale (degrees Celsius). In such cases, an absolute temperature scale is used.

Task

In groups prepare a justification for each of the following statements:

1. $|x| \geq 0$ An absolute value is greater than or equal to zero.
2. $|0| = 0$
3. $|-x| = |x| = x$
4. $|x| + |y| \geq |x + y|$

Inverse of a function

If a function $y = g(x)$ exists and you can find unique solutions $x = f(y)$ for all y, then $x = f(y)$ is the inverse function of $y = g(x)$. This means the function must be one-to-one.

The symbol for the inverse function is f^{-1}. If $y = g(x)$ has inverse $x = f(y)$ then $g^{-1}(x) = f(x)$.

Example 1

Determine the inverse of the function $f(x) = x + 2$.

Solution

$y = f(x)$, $f(x) = x + 2$ making x the subject

$y = x + 2$

$x = g(y)$ $x = y - 2$ take the inverse and write as a function of x.

$f^{-1}(x) = x - 2$

Hence, the inverse of $f(x) = x + 2$ is $f^{-1}(x) = x - 2$

Example 2

You are given the function $f(x) = 3x + 2$. Determine the

a) inverse of $f(x)$, $f^{-1}(x)$.
b) range and domain of $f^{-1}(x)$.

Solution

a) $y = f(x)$ $y = 3x + 2$ $f(x) = 3x + 2$

$x = g(y)$ $x = \frac{y-2}{3}$ $f^{-1}(x) = \frac{x-2}{3}$

b) Domain of $f^{-1}(x)$ is all real numbers.

Range of $f^{-1}(x)$ is all real numbers.

Example 3

Determine the inverse function of $h(x) = x^2, x \geq 0$ (Note this is a one to one function)

Solution

Let $y = x^2$ $\qquad y = f(x) = x^2$

$y = x^2$

$x = \sqrt{y}$ as the domain is strictly positive

$$f^{-1}(x) = \sqrt{x},\ x \geq 0$$

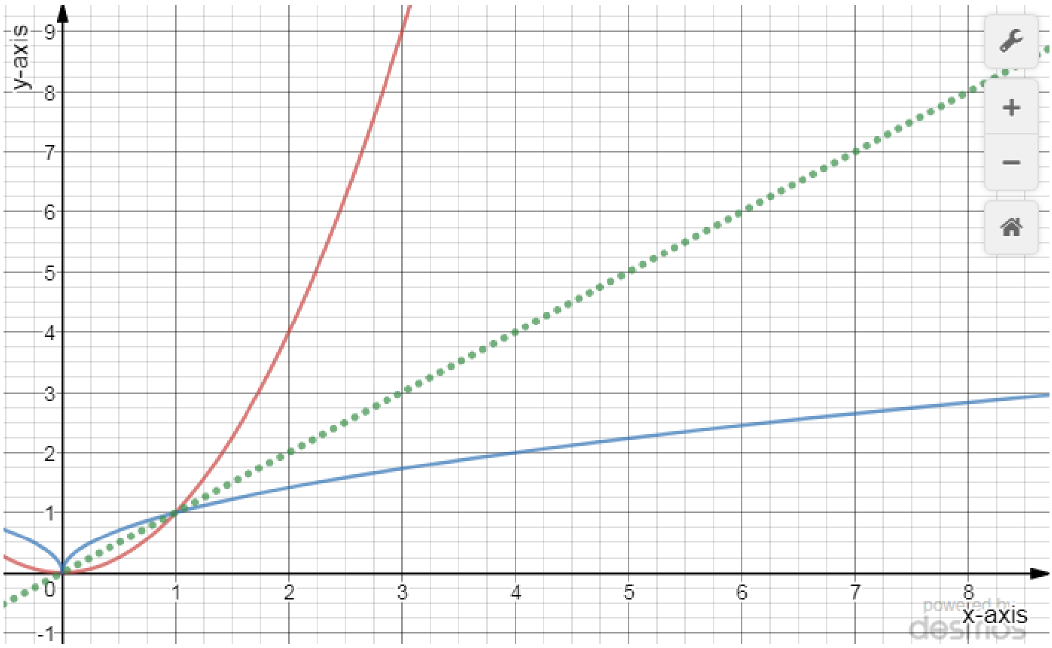
Graphs of inverse functions

Example

On the same axes, draw the graph of the function $f(x) = x^2, x \geq 0$ and its inverse.

Solution

x	0	1	2	3	4	5
$f(x) = x^2$	0	1	4	9	16	25
$f^{-1}(x) = \sqrt{x}$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$



From the graph above, one can identify four relationships between a function and its inverse.

1. The graph $f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$.
2. If a point (a, b) is on $f(x)$ then the point (b, a) exists on $f^{-1}(x)$.
3. The domain and range of the function and its inverse are interchangeable if they both exist.
4. A function must be one-to-one in order to have an inverse function.

Exercise 1.6

1. Determine the inverse of the following functions, stating any conditions on the domain in order to do so. Sketch the function and its inverse.
   a) $f(x) = 3x + 4$
   b) $f(x) = \frac{x}{4}$
   c) $h(x) = \frac{x}{2} + 3$
   d) $f(x) = x^2$

e) $f(x) = x^2 + 2$
f) $f(x) = x^2 + 2x + 1$
g) $f(x) = x^3$
h) $g(x) = \frac{3}{x+2}$
i) $g(x) = \frac{2x}{4-x^2}$
j) $f(x) = \frac{x^2+x+1}{x}$
k) $f(x) = \sqrt{x}$

Composite Functions

A composite function is a function obtained by successively applying two or more functions. The result of a composite function is obtained by applying one function to the results of another.

For example, $f(x) = g(h(x))$, first apply h to x, then apply g to $h(x)$.

Example 1

Given the functions $g(x) = x^2 + 2x + 3$ and $h(x) = 2x + 1$, solve:

a) $f(x) = g(h(x))$
b) $k(x) = h(g(x))$

Solution

a) $f(x) = g(h(x))$
$= g(2x + 1)$
$= (2x + 1)^2 + 2(2x + 1) + 3$
$= 4x^2 + 4x + 1 + 4x + 2 + 3$
$= 4x^2 + 8x + 6$

b) $k(x) = h(g(x))$
$= h(x^2 + 2x + 3)$
$= 2(x^2 + 2x + 3) + 1$
$= 2x^2 + 4x + 6 + 1$
$= 2x^2 + 4x + 7$

Note: In the composition of a composite function, the ordering of the functions is very important since the results are likely to be different.

Example 2

Given the function $g(x) = \sqrt{x-1}$ and $h(x) = x^2$,

Find $f(x) = h(g(x))$ and state the domain and range

Solution

$f(x) = h\left(\sqrt{x-1}\right),\ x \geq 1$; $h(x) = x^2, x \geq 0$; range is all positive reals including zero.

$f(x) = \left(\sqrt{x-1}\right)^2,\ \ x \geq 1$, range is all positive reals including zero.

$f(x) = x - 1,\ \ x \geq 1$, range is all positive reals including zero.

Note: Just considering the composite function solution $f(x) = x - 1$ could be misleading, as it might suggest that the domain and range could be all real numbers.

Composition of inverse functions

Theorem: If a function $f(x)$ is a composition of a function $g(x)$ and its inverse function $g^{-1}(x)$, then $f(x) = x$.

Example 1

Given that$g(x) = x - 2,$ find:

a) g^{-1}(x)
b) $f(x) = g(g^{-1}(x))$

Solution

a) $g(x) = x - 2,$ $\quad y = x - 2$

$$x = y + 2$$

$$g^{-1}(x) = f(y)$$

$$x = f(y) = y + 2$$

Hence $g^{-1}(x) = x + 2$

b) $f(x) = g \cdot g^{-1}(x)$

$$= g(x+2)$$

$$= x + 2 - 2$$

$$f(x) = x$$

Definition

The inverse of a function f is a function f^{-1} that satisfies the property $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

Example 2

Prove that $p(x) = x^4$ and $h(x) = x^{\frac{1}{4}}$ are inverse functions of one another.

Solution

$$p(h(x)) = \left(x^{\frac{1}{4}}\right)^4 = x$$

$$h(p(x)) = (x^4)^{\frac{1}{4}} = x$$

Since $p(h(x)) = h(p(x)) = x$, h and p are inverse functions.

Example 3

Given the function $g(x) = 3x - 11, h(x) = x^2$, find:

a) $f(x) = g(h(x))$
b) $g(1)$
c) $g^{-1}(x)$
d) $g(h(1))$

Solution

a) $g(h(x)) = g(x^2)$
$= 3x^2 = 3x^2 - 11$

b) $g(x) = 3x - 11$
$g(1) = 3 \times 1 - 11$
$g(1) = -8$

c) $g(x) = 3x - 11,$
$y = 3x - 11$
$x = \dfrac{y + 11}{3}$
$g^{-1}(x) = \dfrac{x + 11}{3}$

d) $g(h(x)) = 3x^2 - 11$
$g\big(h(1)\big) = 3 \times 1^2 - 11 = -8$

Exercise 1.7: Work in pairs.

1. Given the functions

$f(x) = 4 - x^2, \quad g(x) = \sqrt{x + 4}$ and $h(x) = \frac{1}{4x},$

Evaluate:

a) $f(g(x))$

b) $f(h(x))$
c) $g(f(1))$
d) $f(g(x^2))$
e) $h(g(x))$
f) $f(g(h(x)))$
g) $g(f(h(x)))$
h) $f(g(h(0)))$
i) $f(g^{-1}(x))$
j) $f(h^{-1}(1))$

2. Given the function $f(x) = x^2 + 3x + 2$ and $g(x) = \sqrt{x+2}$, explain why function $f(g(-3))$ does not exist.

3. Write $f(x) = \sqrt{\frac{1}{x^2+2}}$ as a composition of two functions $g(x)$ and $h(x)$.

4. Show that the following pair of functions are inverse of each other.
   a) $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$.

   b) $f(x) = 3x - 3$ and $g(x) = \frac{x}{3} + 1$

   c) $h(x) = \frac{1}{x^2-1}$ and $f(x) = \sqrt{\frac{1}{x} + 1}$

5. Given a function $g(x) = x^2 + 1$ has a domain $[1, \infty)$, find the domain and range of the inverse function $g^{-1}(x)$.

6. Determine the range and domain of the function $f^{-1}(x)$ given that $f(x) = x^2 - 2$.
7. Write each of the following composite functions as a composition of two functions.

a) $f(h(x)) = \sqrt{x^3} - 1$

b) $g(h(x)) = (3x - 4)^3$

c) $f(g(x)) = \frac{1}{x^2 - 1}$

d) $f(g(x)) = x^2$

8. For each of the following functions, specify the domain of its inverse.
   a) $f(x) = x^2 - 2x + 3$ with domain $[1, \infty)$

   b) $h(x) = |x|$ with domain $(-\infty, \infty)$

   c) $g(x) = 4 - x^2$ with domain $\mathbb{R}^+$

Operations on functions

Functions are operated on like other equations. For instance,

a) $2f(x) = f(x) + f(x)$
b) $f(x) + g(x) = (f + g)(x)$
c) $\frac{f(x)}{g(x)} = f(x) \div g(x)$

Example

Given that $f(x) = 2x^2 + 4$, $g(x) = x^2$ and $h(x) = 2x$, find

a) $f(x) + g(x)$
b) $2h(x) + 3g(x) + f(x)$
c) $(f + g + h)(x)$

Solution

a) $f(x) + g(x) = 2x^2 + 4 + x^2$

$$= 3x^2 + 4$$

b) $2h(x) + 3g(x) + f(x) = 4x + 2x^2 + 4 + 3x^2$
$$= 5x^2 + 4x + 4$$

c) $(f + g + h)(x) = f(x) + g(x) + h(x)$
$$= 2x^2 + 4 + x^2 + 2x$$
$$= 3x^2 + 2x + 4$$

Exercise 1.8: Work in pairs.

1. Given $g(x) = x^2 + 3x + \frac{1}{2}, find$
   a) $2g(x)$
   b) $g(x) + 2g^{-1}(x)$

2. If the function $f(x) = g(x) + 2h(x)$ and $f(x) = x^2 + 7x$ and $h(x) = x^2 + 9x + 2,$ find $g(x)$.

3. If $f(x) = x^2 + 3,\ g(x) = x^2 + 2x + 3$ and $h(x) = \frac{1}{2}$, find
   a) $f(g(x))$
   b) $(f + g)(x)$
   c) $f(x) + 3g(x)$
   d) $(g + f + h)(x)$
   e) $f(g(h(x)))$
   f) $g(h(f(x)))$
   g) $(3h + 2f + 3g)(x)$
   h) $f(g(1))$
   i) $(2f + 3h)(1)$
   j) $f(0) + 2h(1) + 3h(3)$

UNIT 2: TRIGONOMETRY

Introduction

The word trigonometry is derived from two Greek words: *trigono* and *metry* that means a triangle and measurement, respectively. In our basic mathematics, we discussed various types of triangles. The measurements of a triangle that can be made include the dimensions of the sides and the angles at which these meet.

In general, we define trigonometry as a branch of mathematics that deals with the relationship of the dimensions of the sides of a triangle and their angles and the functions of the angles.

Trigonometric Ratio Identities

Task 1

In groups study and discuss the figure below of a unit circle center O(0, 0) with a right triangle OQP such that, its vertex P is on the circumference and angle $POQ = \theta^0$. Answer the following questions:

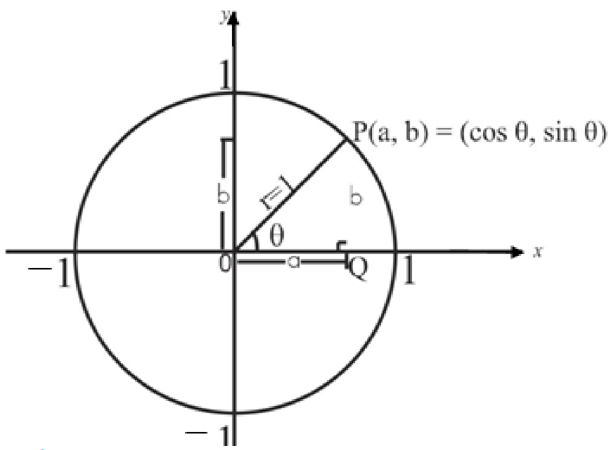


Figure 2.1

a. Write an expression for ;

   i. Sine θ

   ii. Cosine θ

   iii. Tangent θ

b. What are complementary angles?

c. Identify the pair of complementary angles in figure 2.1 above.

d. Write general expressions for;

   i. $\sin(90 - \theta)^0$

   ii. $\cos(90 - \theta)^0$

e. From the solution in a and d above what do you infer about cosine and sine of complementary angles.

i. Draw a line $x = 1$ that is parallel to y-axis and meets the x-axis at point A(1, 0). Extend the side *OP to* meet the line $x = 1$ at *B*. Using similarity show $\tan\theta = \frac{\sin\theta}{\cos\theta}$

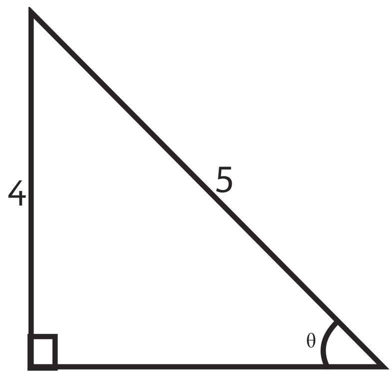
Exercise 2.1: Individually

1. In each of the following triangles, find the values of the given ratios. (Leave your answer as a fraction).

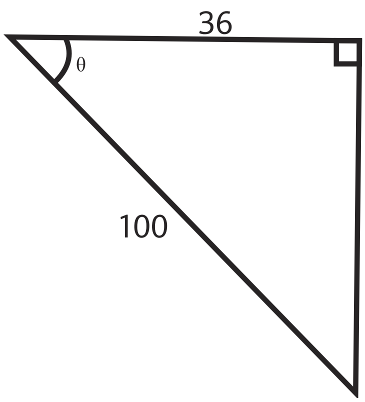
   a) $\tan\theta$

   b) $\sin\theta$

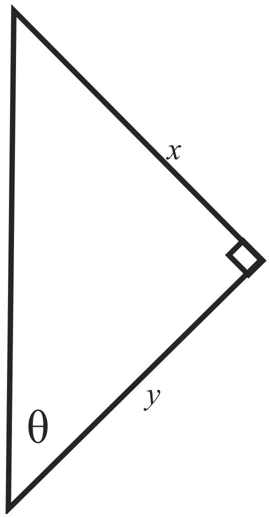
   c) $\cos\theta$

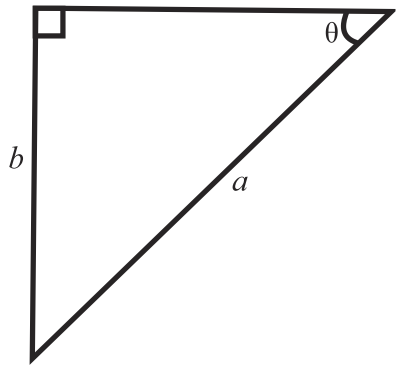


a)



c)

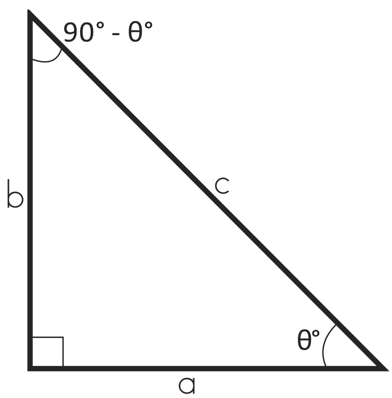
b) 

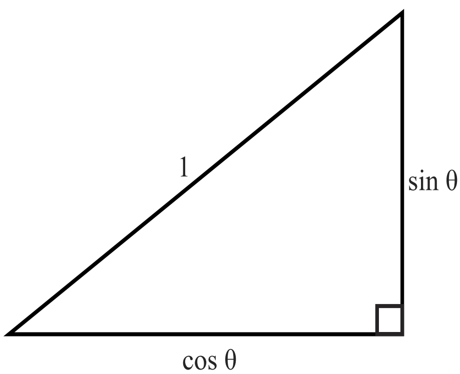
d) 

2. If x is an acute angle in a right-angled triangle, $\tan x = \frac{12}{5}$, find the value of:
   a) $\sin x$
   b) $\cos x$
3. If y is an acute angle in a right-angled triangle whose $\tan y = 5$, find, in surd form:
   a) $\cos y$
   b) $\sin y$

Task 2

Study figure 2.2 below and compare the two triangles. Use them to derive expressions for $\cos\theta$, $\sin\theta$ and $\tan\theta$





There are three additional trigonometric functions, that are the reciprocals of the three met thus far:

Cosecant is the reciprocal of Sine.

$$Cosecant\ \theta = \frac{1}{\sin\theta} \quad \text{or} \quad Csc\theta = \frac{1}{\sin\theta}$$

Secant is the reciprocal of Cosine.

$$Secant\ \theta = \frac{1}{\cos\theta} \quad \text{or} \quad Sec\ \theta = \frac{1}{\cos\theta}$$

Cotangent is the reciprocal of tangent.

$$Cotangent\ \theta = \frac{1}{tangent\ \theta} \quad \text{or} \quad Cot\ \theta = \frac{1}{\tan\theta}$$

Special Acute Angles in Trigonometry

The right-angled isosceles triangle and equilateral triangle are very important in day-to-day life. They are mainly used in construction. It is possible to obtain exact values of the trigonometric functions for $45°$, $60°$ and $30°$ from such triangles. The values obtained can be applied in all trigonometric problems.

Task 3

Determine $\cos\theta$, $\sin\theta$ and $\tan\theta$ for the isosceles right angled triangle in figure 2.3.

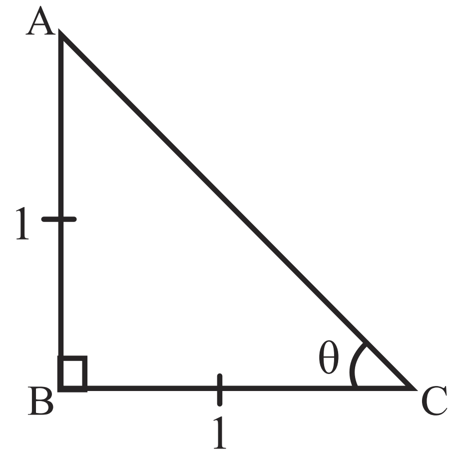
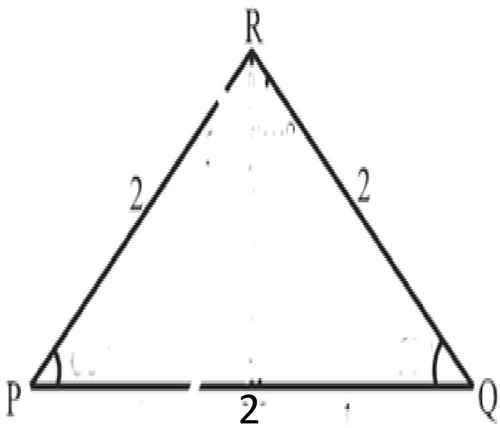


Figure 2.3

Hint: Use Pythagoras' theorem to find AC. Use angle sum of a triangle to find angle θ.

Task 4

The figure below shows an equilateral triangle $\Delta\ PQR$.
$PQ = QR = RP = 2\ units$. Determine $\cos\theta, \sin\theta$ and $\tan\theta$ and for $90° - \theta$.



Hint: Drop a perpendicular from R to PQ. Use Pythagoras' theorem to find the perpendicular height. Use angle sum of a triangle to find angle θ.

Exercise 2.2: Work in pairs.

1. Determine the complementary angles for the angles below:
   a) $30°$
   b) $60°$
   c) $47°$
   d) $45°$
   e) $37°$
   f) $20°$
   g) $74°$
   h) $90°$
2. Find the acute angles x or y in each of the following questions.
   a) $\cos x° = \sin 45°$
   b) $\sin 20° = \cos y°$
   c) $\sin 40° = \cos x°$
   d) $\cos 20° = \sin y°$
   e) $\sin 40° = \cos x°$
   f) $\sin 25° = \cos y°$

3. Using your results from tasks 3 and 4, simplify the following without using calculators. Leave your answer as a rational number.
   a) $\sin 30°$
   b) $\sin 45° \cos 45°$
   c) $4 \cos 30° \sin 45°$
   d) $\dfrac{\cos 45° + 2 \sin 30°}{\tan 45°}$
   e) $\dfrac{(\cos 45° + \sin 30°)(\tan 30°)}{\cot 30°}$
   f) $\dfrac{\cot 30° \sec 45°}{\cot 60° \sin 30°}$
   g) $\dfrac{(\cot 30° + \sec 30°)}{\sin 45° + \cos 45°}$
   h) $\dfrac{\cos^2 45° + \cot^2 60°}{\tan 30°}$
   i) $\dfrac{\cos 30° \cdot \cot^2 60° \cdot \sec^2 60°}{\cot^2 30° + \tan^2 30°}$

4. The angle made by the hydraulic machine pulleys and its horizontal base is $30°$. The vertical distance from the base to the vertex is 10m. Without using a calculator, find:
   a) The horizontal distance between the tips of the pulley.
   b) The length of the arms of the pulley.

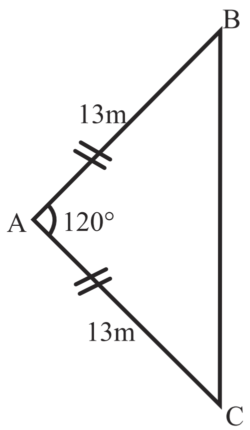
Hint: draw a diagram

5. The angles at the vertex of a bulb glass holder whose shape is a cone is $90°$. If its slanted height is $3\sqrt{2}$ m , find
   a) The diameter of the cone.
   b) The height of the cone.

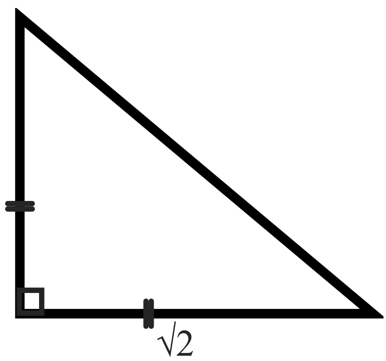
Hint: draw a diagram

6. Show that $\sin^2 30° + \cos^2 30° = 1$
7. Without using a calculator, calculate the missing dimensions in the following triangles.

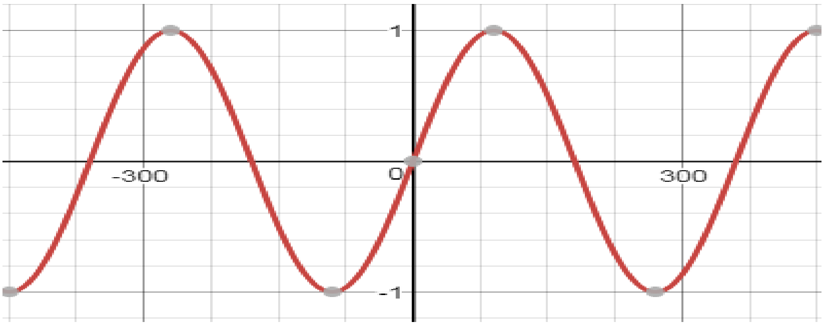
a)



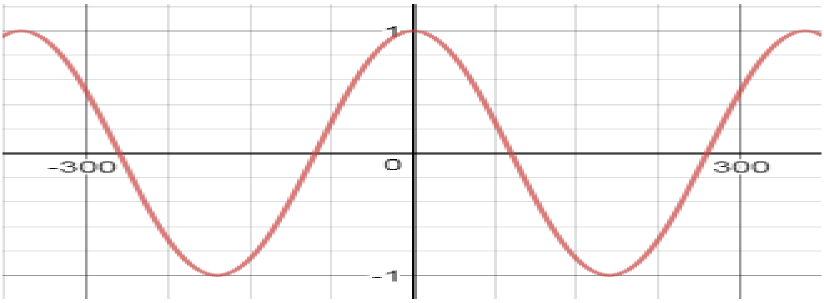
b)



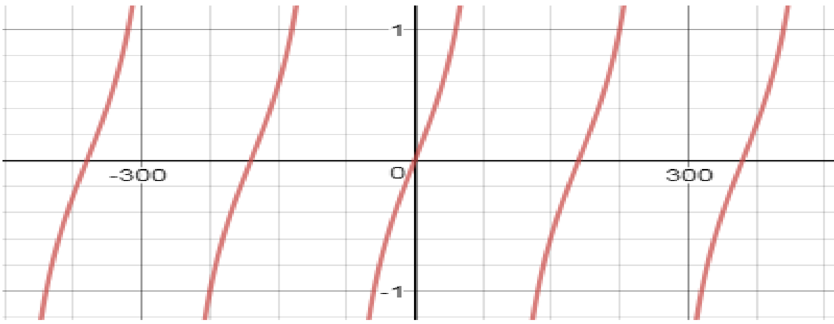
Finding Trigonometric values for non-acute angles



$y = \sin\theta$



$y = \cos\theta$



$y = \tan\theta$

Trigonometric functions are periodic, which means there are multiple values of for any given .

Task

Given that $\sin 30° = \frac{1}{2}$, use the graph to find:

a) $\cos 60°$

b) $\sin 150°$

c) $\sin 210°$

d) $\sin 330°$

e) $\sin(-30°)$ f) $\sin(-150°)$

Exercise 2.3: Work in pairs.

1. Leaving your answer in surd form, simplify the following expressions.
   a) $\sin -45°$
   b) $\cos -30°$
   c) $\tan 60°$
   d) $\cos 45°$
   e) $\cot -60°$
   f) $\sec 30°$
2. Find the value of the following trigonometric expressions without using a calculator or tables.
   a) $\cos 150°$
   b) $\tan 210°$
   c) $\sec 240°$
   d) $\csc 240°$
3. Find the trigonometric function values using properties of special angles.
   a) $\tan(-120°)$
   b) $\cot(-225°)$
   c) $\sec(-150°)$
4. Find the values of the following leaving your answer as a fraction
   a) $\cos 750°$
   b) $\csc -420°$
   c) $\sin(-600°)$
   d) $\cos(510°)$
5. Evaluate and leave your answer in surd from.
   a) $\cos 30° + \cos 330°$
   b) $\sin 420 + 2\sin(-60°)$
   c) $3\cot 120° + 4\sec(-150°)$
   d) $\frac{3\cos 30° \cdot \sin 45°}{\cot 120°}$
   e) $\frac{\cot 150° \sec 225°}{2\cot(-60°) \sin 390°}$
   f) $\frac{\cot 30° + \sec 150°}{\sin 405° \cos 405°}$

g) $\frac{\cos^2 45° \cdot \cot^2 120° \cdot \sec(-240°)}{\cot^2(1830)° + \sin^3(-210°)}$

Pythagoras' Theorem and Derived Trigonometric Identities

Using Pythagoras' theorem and the unit circle, it is possible to derive several other trigonometric identities.

Task

The figure below shows a unit circle and a right triangle OQB. In groups of four students study it and answer the questions that follow

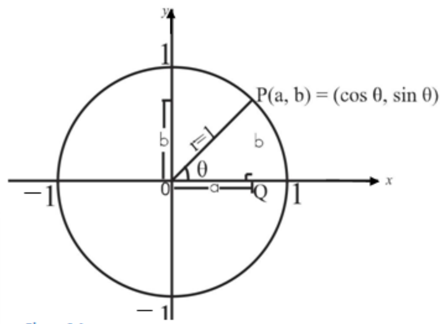


Figure 2.1

i. Use the triangle OQP to prove the identity $\cos^2\theta + \sin^2\theta = 1$

ii. Use the identity $\cos^2\theta + \sin^2\theta = 1$ to show that

  a. $1 + \tan^2\theta = \sec^2\theta$

  b. $\cot^2\theta + 1 = \csc^2\theta$

The above identities can be used to prove several other trigonometric relationships.

Example 1

Show that $\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = 2\sec x.$

Solution

To show the left hand side (LHS) and the right hand side (RHS) of the equation are equal, select one side of the equation and simplify it.

$$2\sec x = \frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x}$$

LHS RHS

Leaving LHS and simplifying RHS.

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x}$$

$$= \frac{(\cos x)\cos x + (1+\sin x)(1+\sin x)}{(1+\sin x)\cos x}$$

$$= \frac{\cos^2 x + 1^2 + 2\sin x + \sin^2 x}{(1+\sin x)\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x + 1 + 2\sin x}{(1+\sin x)\cos x}$$

$\cos^2 x + \sin^2 x = 1$

$$= \frac{1+1+2\sin x}{(1+\sin x)\cos x}$$

$$= \frac{2+2\sin x}{(1+\sin x)\cos x}$$

$$= \frac{2(1+\sin x)}{(1+\sin x)\cos x}$$

$$= \frac{2}{\cos x} = 2\sec x$$

So RHS is equal to LHS as required.

Example 2

Show that $2\tan\theta = \frac{\cos\theta}{1-\sin\theta} - \frac{\cos\theta}{1+\sin\theta}$.

Solution

Simplifying RHS,

$$\frac{(1+\sin\theta)\cos\theta - (1-\sin\theta)\cos\theta}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{\cos\theta + \cos\theta\sin\theta - \cos\theta + \cos\theta\sin\theta}{1-\sin^2\theta}$$

$$= \frac{2\cos\theta\sin\theta}{1-\sin^2\theta}$$

$$= \frac{2\cos\theta\sin\theta}{\cos^2\theta}$$

$$= \frac{2\sin\theta}{\cos\theta}$$

$$= 2\tan\theta$$

RHS is equal to LHS as required.

Exercise 2.4: Discuss in groups and prepare to present your findings to the class.

Select three from the following identities and prove them.

a) $\sec\theta = \frac{\cos\theta}{1-\sin\theta} - \tan\theta$

b) $\frac{\csc\beta\cos\beta}{\tan\beta+\cot\beta} = \cos^2\beta$

c) $\frac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

d) $\tan\theta\sin\theta - \sec\theta = \cos\theta$

e) $\frac{\tan^2\theta-1}{\tan^2\theta+1} = 1 - 2\cos^2\theta$

f) $\sin^2\theta = \frac{\tan^2\theta}{\tan^2 x+1}$

g) $\sin^3\beta = \sin\beta - \sin\beta\cos^2\beta$

h) $\frac{3+\sin\alpha}{1-\sin\alpha} = \frac{\sin^2\alpha+4\sin\alpha+3}{\cos^2\alpha}$

i) $\frac{\cos\gamma}{1+\sin\gamma} = \frac{1-\sin\gamma}{\cos\gamma}$

j) $\frac{1}{\tan x\cos x} = \frac{1+\tan^2 x}{\tan x}$

k) $\frac{\sin^4\theta-\cos^4\theta}{\sin^2\theta-\cos^2\theta} = 1$

l) $(\cos\theta+\sin\theta)^2 + (\sin\theta-\cos\theta)^2 = 2$

m) $(\csc\theta+\cot\theta)(\csc\theta-\cot\theta) = 1$

n) $\sec x \csc x = \tan x + \cot x$

o) $\frac{\sin^2 x}{1+\cos^2 x} = \frac{\sec x + \cos x}{\sec x + \cos x}$

p) $\frac{1-\sin\theta}{1+\sin\theta} = \sec\theta - \tan^2\theta$

q) $\sec\theta = \sin\theta(\cot\theta + \tan\theta)$

Compound angle formulas for Trigonometry

A compound angle is a sum, difference, product or quotient of a variable angle.

For instance:

a) Sum and difference of angle identities
b) Double angle identities
c) Half angle identities

In this section, we assume x and y are acute angles greater than 1°, though these formulas are true for both positive and negative angles. Similarly $x + y, x - y$ must be acute angles. These identities include:

a) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
b) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
c) $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$$where\ x + y \neq (k - \frac{1}{2})180°$$

d) $\sin 2x = 2 \sin x \cos x$
e) $\cos 2x = \cos^2 x - \sin^2 x$
f) $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

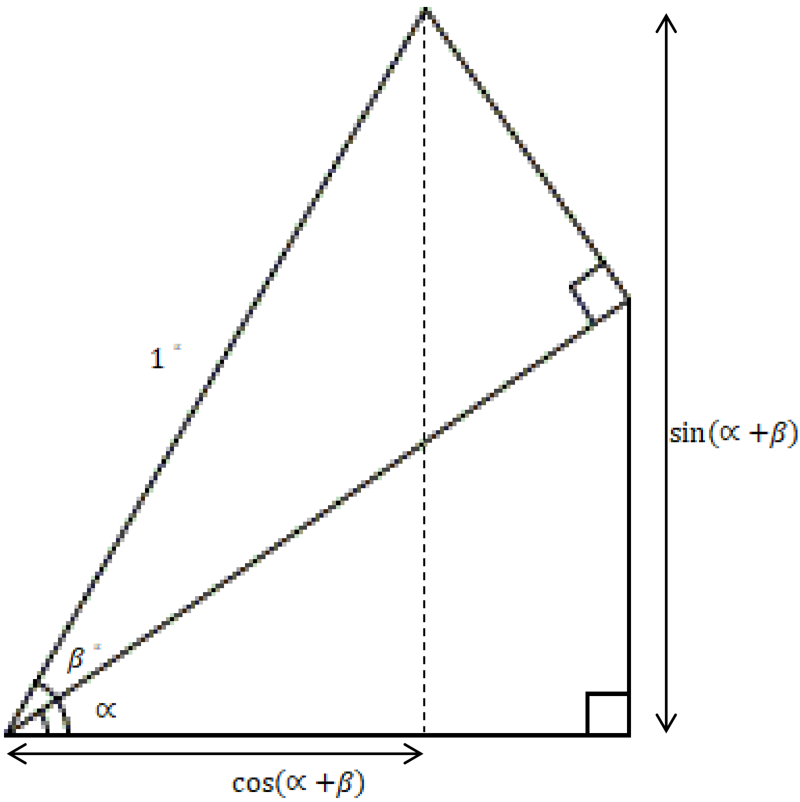
$$where\ x \neq \left(\frac{1}{2}k + \frac{1}{4}\right)180°$$

g) $2\cos^2 x = 1 + \cos 2x$
h) $2 \sin 2x = 1 - \cos 2x$

Sum and Difference of angle Identities

Task 3: Work in groups.

Use a large copy of the diagram below to derive $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$.



Task 4

Replace β with $-\beta$ in your expressions for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ to derive expressions for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$

Deriving Double angle identities

Double angle identities are derived from the sum and difference angle identities. As illustrated in example below.

Task 5

Replace β with α in your expressions for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ to derive expressions for $\sin(2\alpha)$ and $\cos(2\alpha)$

Deriving Half Angle identities

Half angle identities are derived from double angle identities. Look at the example below.

Example 1

Prove the identity $2\sin^2\theta = 1-\cos 2\theta$.

Solution

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos 2\theta = 1-\sin^2\theta - \sin^2\theta$$

$$\cos 2\theta = 1-2\sin^2\theta$$

$$2\sin^2\theta = 1-\cos 2\theta$$

Other exact values of trigonometric functions

Angles that are multiples of $7\frac{1}{2}°$ are special angles since they can be constructed by use of a pair of compasses and a straight edge. The values for the trigonometric functions of these angles can be determined exactly without using a calculator.

Example 2

Find the value of $\cos 15°$ without using a calculator.

Solution

Using double angle identity

$$2\ cos^2 A = 1+\cos 2A$$

Hence $2\ cos^2 15° = 1+\cos 30°$

But $\cos 30° = \frac{\sqrt{3}}{2}$ hence

$$2\ cos^2 15° = 1 + \frac{\sqrt{3}}{2}$$

$$cos^2 15 = \frac{1}{2} + \frac{\sqrt{3}}{4} = \frac{2+\sqrt{3}}{4}$$

$$\cos 15° = \sqrt{\frac{2+\sqrt{3}}{4}} == \frac{\sqrt{2+\sqrt{3}}}{2}$$

Task

In groups, complete the table below with exact values

$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0			
15			
30			
45			
60			
75			
90			

Exercise 2.5: Work in groups. Present your findings to the class.

1. Using the identities;
   $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
   $\cos(A \pm b) = \cos A \cos b \mp \sin A \sin B$
   Prove each of the following identities.
   a) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

b) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$ for $A = B \neq \left(k + \frac{1}{2}\right)180°$

c) $\sin 2A = 2 \sin A \cos A$

d) $\tan 2A = \frac{2 \tan A}{1 - \tan A}$ for $A \neq \left(\frac{1}{2}k + \frac{1}{4}\right)180°$

e) $2\cos^2 A = 1 + \cos^2 2A$

2. If $2\cos^2\theta = 1 + \cos 2\theta$, $2\sin^2\theta + 1 - \cos 2\theta$ and $t = \tan\frac{1}{2}\theta$ show that;

   a) $\sin\theta = \frac{2t}{1+2t}$

   b) $\cos\theta = \frac{1-t^2}{1+t^2}$

   c) $\tan\theta = \frac{2t}{1-t^2}$

3. Find an expression for $\sin 3x$ in terms of $\sin x$.

4. Prove that $\cos 3x = 4\cos^3 x - 3\cos x$.

5. Evaluate the following without using a calculator.

   a) $\tan 75°$

   b) $\sin 22\frac{1}{2}°$

   c) $\frac{\tan 35° \cos 30°}{\sin 75°}$

   d) $\frac{\cos 150° \sin 390°}{\tan 30°}$

6. Solve the equations.

   a) $\cos 2x = \sin x$ for $-180° \leq x \leq 180°$

b) $\sin 2x = \sin x$ for $0° \leq x \leq 360°$

c) $\cos 2x = \cos x$ for $-180° \leq x \leq 180°$

Relationship between Sum and Difference of Angles with the products of Trigonometric Ratios

Task

In pairs, use

i. $\sin(\theta + \beta) = \sin\theta\cos\beta + \cos\theta\sin\beta$

ii. $\sin(\theta - \beta) = \sin\theta\cos\beta - \cos\theta\sin\beta$

iii. $\cos(\theta + \beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$

iv. $\cos(\theta - \beta) = \cos\theta\cos\beta + \sin\theta\cos\beta$

To derive identities for the products $\sin\theta\cos\beta$, $\cos\theta\cos\beta$ and $\sin\theta\sin\beta$

Hint: Add i and ii. Add iii and iv.

Task - Simpson's Formulas

Derive Simpson's formulas by substituting $\theta = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$

in the identities derived in the previous task

$$\theta + \beta = \frac{x+y}{2} + \frac{x-y}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x$$

$$\theta + \beta = x$$

$$\theta - \beta = \frac{x+y}{2} - \frac{x-y}{2} = \frac{x+y-x+y}{2} = \frac{2y}{2} = y$$

$$\theta - \beta = y$$

Simpson's formulas are given below.

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \cos y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Exercise 2.6: Individually.

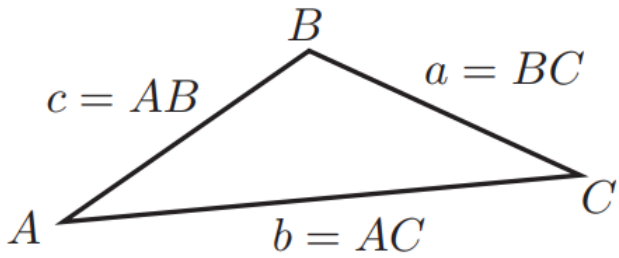
1. Simplify:
    a) $\frac{\sin A - \cos B}{\sin A + \cos B}$
    b) $\tan(A + 45°) + \cot(A + 45°)$
2. Factorise:
    a) $\tan A - \sin A$
    b) $\cos 4\beta + \cos 5\beta + \cos 6\beta$
    c) $\cos^2 A - \sin^2 A$
3. Without using tables or a calculator, evaluate:
    a) $\sin 15° + \cos 15°$
    b) $\cos 22\frac{1}{2}° + \cos 22\frac{1}{2}°$
    c) $\sin 75° - \sin 15°$
    d) $\cos 105° - \cos 15°$
4. Express each of the following in terms of $\cos A$
    a) $\frac{\sin\frac{A}{2} + \cos\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}}$
    b) $\cos 4A$
    c) $\tan\frac{A}{2}$
    d) $\tan 3A$

The sine rule and cosine rule

These rules are used to calculate the length of sides of a triangle and the angles of a triangle. A triangle has three sides and three angles. The angles and vertices of a triangle are marked with capital letter while the lengths are labelled with small letters of the opposite vertices.

Task **- The sine rule**

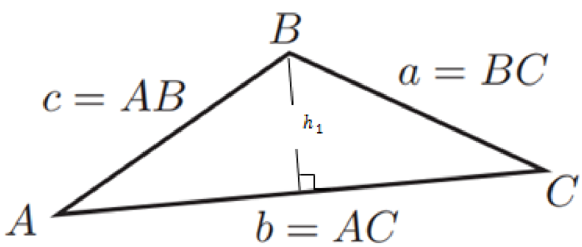
The figure below shows triangle *ABC*.



a. Drop a perpendicular line from vertex B to the opposite side of the triangle (AC) write two expressions for the length of the perpendicular.
b. Drop a perpendicular line from vertex A to the opposite side of the triangle (BC) write two expressions for the length of the perpendicular.
c. Hence, derive an expression relating the sines of the angles with the lengths.

Task – the cosine rule

In groups, derive the cosine rule $a^2 = b^2 + c^2$ +2ab Cos A for a triangle *ABC*



Hint: Use Pythagoras' theorem in the two right-angled triangles created by the perpendicular from B to AC.

Example

Solve triangle *ABC*, given that *AB* = 42cm, *BC* = 37cm and *AC* = 26cm.

Solution

We are given three sides of the triangle and so the cosine rule can be used.

Writing a = 37, b = 26 and c = 42 we have $a^2 = b^2 + c^2 - 2bc\cos A$.

Substituting $37^2 = 26^2 + 42^2 - 2(26)(42)\cos A$

$$Cos\ A = \frac{26^2 + 42^2 - 37^2}{2(26)(42)} = 0.4904 \quad \text{to 4d.p.}$$

A = cos^{-1} 0.4904 = 60.6^0 to1.d.p.

The sine rule can be used to derive B (37.8°) or C (81.6°), and the angle sum of the triangle to find the third angle.

Application of Trigonometry

Throughout its early development, trigonometry was often used as a means of indirect measurement, e.g. determining large distances or lengths by using measurements of angles and small, known distances. Today, trigonometry is widely used in physics, astronomy, engineering, navigation, surveying, and various fields of mathematics and other disciplines.

Exercises 2.7: Work in groups.

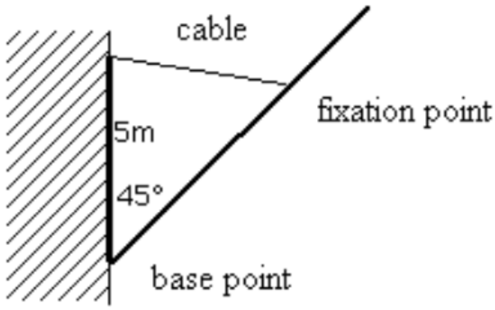
1. The angle of elevation of the top of a tower from the ground is 30°. At a position 24 metres closer to the tower, the angle of elevation is 45°. Determine the height of the tower.

   Hint: Draw a diagram

2. Two planes depart from the same point, each in a different directions. The directions form an angle of 45°. The velocity of the first plane is 300 km/hour, the velocity of the second is 600 km/hour. Determine the distance between them after one and a half hours.

   Hint: Draw a diagram

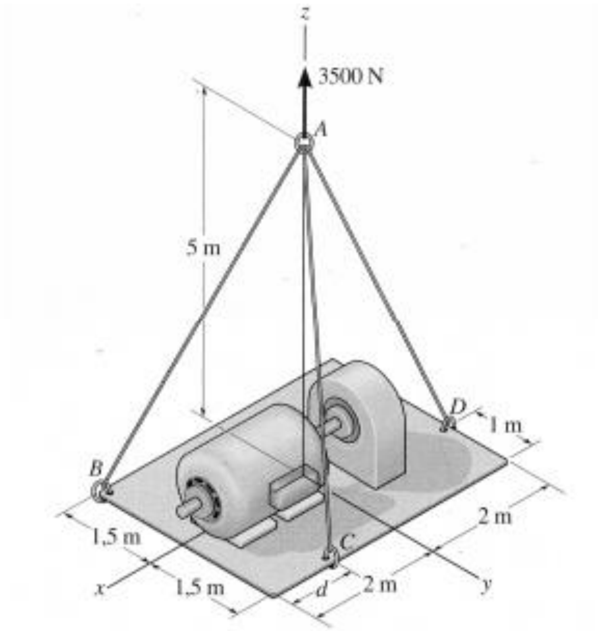
3. A flagpole is fastened to a wall at an angle of 45°. Five metres above the base point of the pole in the wall, there is a cable fixed to the wall with a length of 3.6 m. How far along the pole is the fixation point?



4. A plane sails north and sees a tower on a bearing of 040°. After sailing 20 km, the bearing has increased to 080°. Determine at both positions the distance from the plane to the tower.

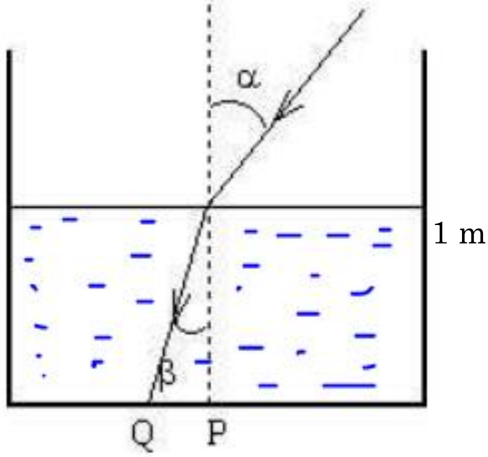
   Hint: Draw a diagram

5. The diagram shows a mechanical motor. Determine the angle between the ropes AC and AD (d = 1 m).

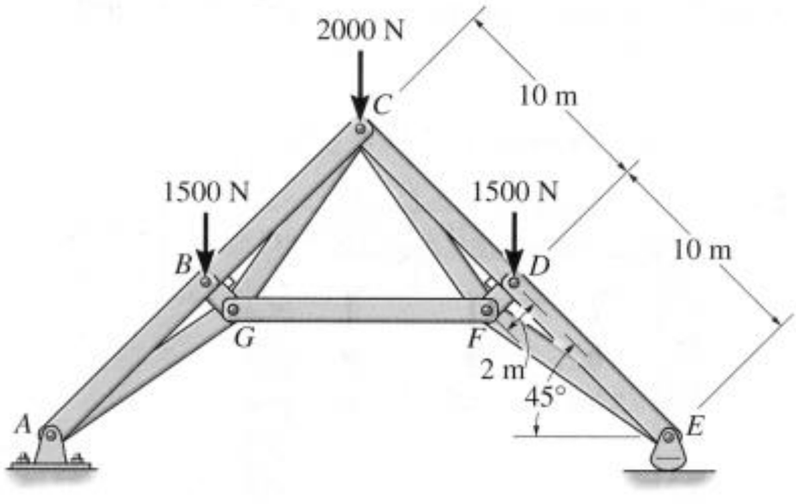


6. An incident ray is a ray of light that strikes a surface. The angle between this ray and the perpendicular or normal to the surface is the angle of incidence (α). The refracted ray or transmitted ray corresponding to a given incident ray represents the light that is transmitted through the surface. The angle between this ray and the normal is known as the angle of refraction (β). The relationship between the angles of incidence and refraction is given by Snell's Law expressed as $\frac{\sin \alpha}{\sin \beta}$ = refractive index.

   In the figure below a ray of light strikes an air-water interface at an angle of 30 degrees from the normal (α = 30°). The relative refractive index for the interface is $\frac{4}{3}$. At what distance from PQ does the ray of light hit the bottom, if the water is 1 m deep.



7. The figure below shows a sketch of a house roofing bars.



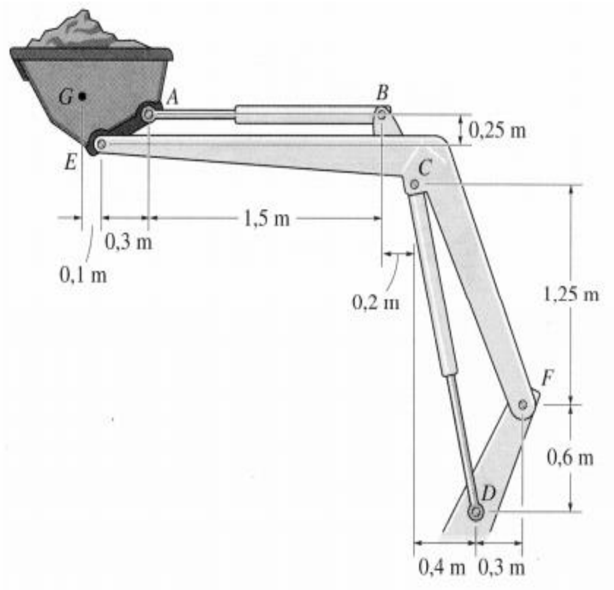
Calculate:

i. the angle between FE and ED.

ii. the vertical height C from the ground if the roofing starts 8m above the ground.

In mechanics you will deal with forces applied at designed distances and on specified angles to make the work easier. The angles at which various parts meet must be calculated and drawing designs made. In the following questions 8 and 9 such situations are sketched. We will confine to the calculation of angles between various bars.

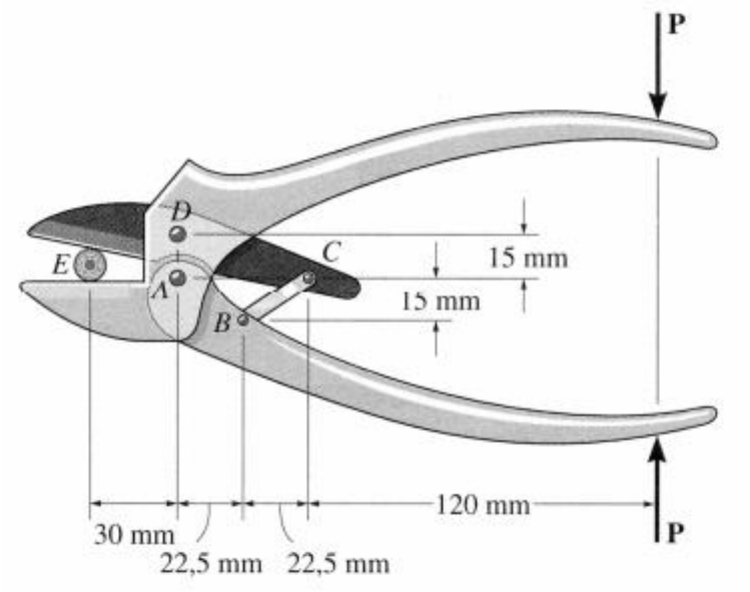
8. The figure below shows part of a working hydraulic machine. If the indicated distance are horizontal and perpendicular distances, study the diagram and answer the questions that follow.



a. Calculate the angle between line CD and DF.

b. Calculate the angle between line AE and line EC.

9. The figure below shows a pair of pliers being used by forces P and Q to cut a wire into two parts.



Calculate

a. Distance BC and CD.

b. Angle between BC and CD.

UNIT 3: CALCULUS 1 (DIFFERENTIATION)

Introduction

The process of determining the measure of slope or steepness of a curve is called differentiation. The measure of steepness of the curve of a function of *x*, f*(x)* at a point Q is the gradient of the tangent to the curve at *Q*. The gradient tells you about the rate of change of the function.

Differential calculus involves determining the gradient function of functions. A function defines how two quantities relate with each other. In differentiation, we will investigate how the change in two quantities relates at a given instance.

From the first principal we learnt that gradient of a curve at a given point is the same as the gradient of the tangent at that point. A tangent is a line that touches a curve at only one point.

For the function y = *f(x)*, the function that represents its gradient at any general point *x* is called the derivative (or differential function) of the function. The symbol for the derivative is $f' \ or \ \frac{dy}{dx} \ or \ f'(x)$

The derivative of a function, $y = f(x)$ is hence $f'(x)$. $f'(x)$ is the equation of the tangent to the curve as shown in figure 3.1 below.

On solving for the general point *x* of the function $f'(x)$ we obtain gradient function or derivative.

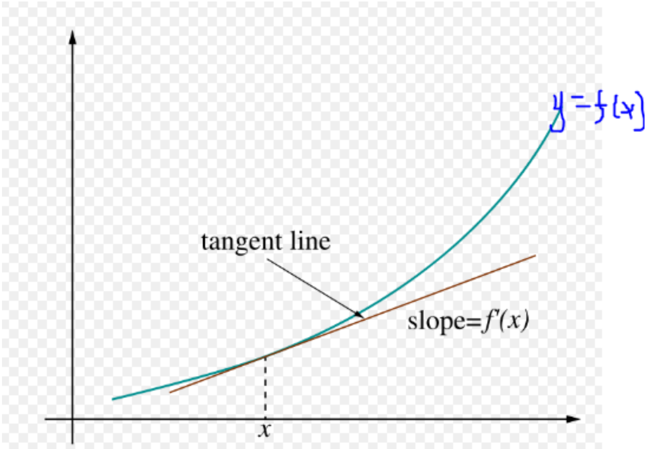


Figure 3.1

As a curve changes the movement, direction of the gradient of the function changes.

Example 1

Study figure 3.2 below. The figure has graph of a function $f(x)$ with points $A(-0.5, f(-0.5))$, $B(0.5, f(-0.5))$ and $C(1.5, f(1.5))$

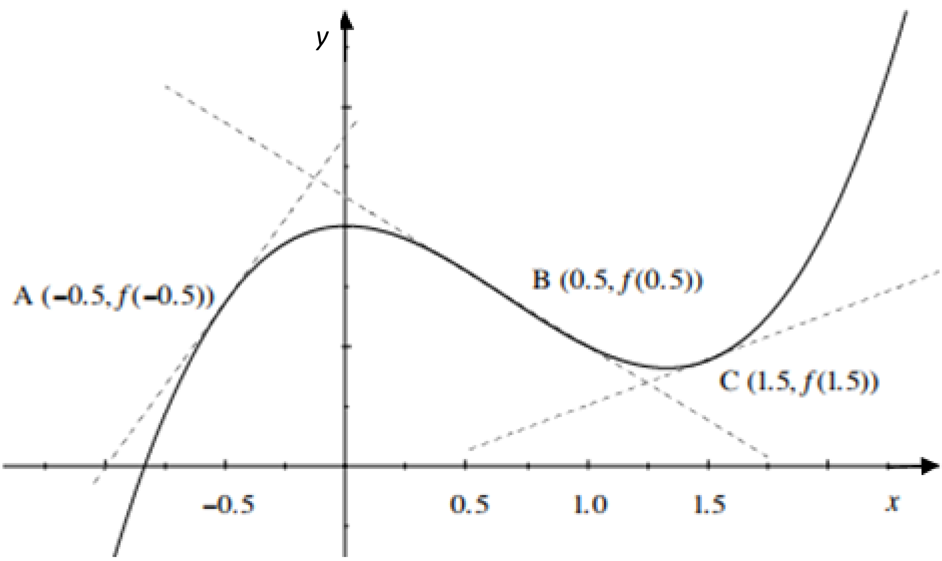


Figure 3.2

a. What is the relationship between the dotted lines and the gradient of the curve?
b. What is the sign of the gradient at points $A(-0.5, f(-0.5))$, $B(0.5, f(-0.5))$ and $C(1.5, f(1.5))$?

Solution

From the diagram it is clear that the gradient at points $A(-0.5, f(-0.5))$, $B(0.5, f(-0.5))$ and $C(1.5, f(1.5))$ is represented by the dotted lines at these points.

At $A\ (-0.5, f(-0.5))$ the gradient is positive, i.e. the function is increasing.

At $B\ (0.5, f\ (0.5))$ the gradient is negative, i.e. the function is decreasing.

At $C\ (1.5, f\ (1.5))$ the gradient is positive i.e. the function is increasing.

The table below shows various ways in which derivative of functions can be expressed in different ways as shown in table 3.1 below.

Expression of a function	Expression of derivative functions
$f(x)$	$f'(x) or \frac{dy}{dx}$
f	$f'\ or\ \frac{d(f)}{dx}$
y	$y'\ or\ \frac{dy}{dx}$
$y(x)$	$y'(x)\ \ or\ \frac{dy(x)}{dx}$

If a function of x is expressed as $f(x) = ax^n$. Its first derivative function is expressed as $f'(x) = \ nax^{n-1}$

Example 2

Find the value of $f'(x)$ given that $f(x) = 2x^5$

Solution

$f(x) = 2x^5$

If $f(x) = ax^n$ then $f'(x) = nax^{n-1}$

$f'(x) = 10x^4$

Example 3

Determine the value of (i) $f'(x)$ and (ii) $f'(2)$ for the functions

a. $y = 6x^2$
b. $f(x) = 24x^3$

Solution

a. $y = 6x^2$,
$y' = f'(x) = 12x$
$f'(2) = 12x = 12 \times 2 = 24$

b. $f(x) = 24x^3$
$f'(x) = 72x^2$
$f'(2) = 72 \times 2^2 = 72 \times 4 = 288$

Exercise 3.1: Work in pairs.

1. Find the derivative of the functions below.

a) $y = 2x$

b) $y = 3x^3$

c) $f(x) = 4x^3$

d) $f(x) = 20x^4$

e) $f(x) = \frac{2}{x^2}$

2. Find the value of $f'(x)$ in each of the following functions.

   a) $f(x) = 20x$

   b) $f(x) = 30x^3$

   c) $f(x) = 4x$

   d) $\frac{y}{x} = 3x^2$

3. a) Find the value of $f'(2)$ if $f(x) = 3x^4$

   b) Find the value of $f'(-3)$ if $y = 3x^7$

   a) Solve for $f'(3) + f'(2)$ if $f(x) = 4x^3$

Differentiation of Polynomials

A polynomial is a function comprising powers (greater than or equal to zero) of a variable. The degree (n) of a polynomial is the highest power of the variable.

General formula of a polynomial function of x is $p(x)$ such that,

$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \ldots + a_0x^0$ $a_0$where n is a non-negative integer and $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are constants.

Example

Find the derivative function of a polynomial

$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \ldots$

Solution

The derivative function of a polynomial $p(x)$ is the sum of the derivatives of each term of the function.

General derivative of $p(x) = p'(x)$

$$p'(x) = a_n n x^{(n-1)} + a_{n-1}(n-1)x^{(n-2)} + a_{n-2}(n-2)x^{(n-3)} + a_{n-3}(n-3)x^{(n-4)} + \cdots + a_1$$

Task

Find the derivative of the following functions

a) $y = 2x^3 + 3x^2 + 3x + 3$
b) $f(x) = x^4 + \frac{1}{3}x^3 + 20x^2 + 3$

Task

Use the function $f(x) = 3x^4 + 4x^2 + 40x$ to find the value of;-

a) $f'(x)$
b) $f'(0)$
c) $f'(20)$

Solution

a) $f(x) = 3x^4 + 4x^2 + 40x$

$f'(x) = 12x^3 + 8x + 40$

b) $f'(x) = 12x^3 + 8x + 40$

$f'(0) = 0 + 0 + 40$

$f'(0) = 40$

c) $f'(x) = 12x^3 + 8x + 40$

$f'(20) = 12 \times 20^3 + 8 \times 20 + 40$

$f'(x) = 96\ 200$

Example 4

Determine the derivative of the following series

a) $g(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$

b) $h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

c) $h(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} \ldots$

Solution

a) if $g(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$

$g'(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

b) $h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$h'(x) = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3.2!} + \frac{4x^3}{4.3!} + \cdots$

$h'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Note that when the term increases $h(x) \to h'(x)$

c) $h(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} \ldots$

$h'(x) = 1 + x + x^2 + x^3 + x^4 \ldots$

Exercise 3.2: Work in pairs.

1. Find the derivative of the following functions

a) $f(x) = x^3 + 4x^2$

b) $f(x) = x^3 + \frac{1}{x^4} + \frac{1}{x^5}$

c) $f(x) = \frac{x^3+x^2+x}{x}$

d) $f(x) = x^4 + \frac{1}{2}x^3 + \frac{1}{2}x$

2. Find the derivative of the following functions

a) $f(x) = (x^2 + x^3)(x^4 + 6x^5)$

b) $f(x) = \frac{1}{2x}(x^4 + 3x^3)$

c) $f(x) = 2x^{-4} + 2x^3 + 5x^2$

d) $f(x) = \frac{x^4-16}{x+2}$

e) $f(x) = \frac{4x^2-81}{2x+9}$

f) $g(x) = x^7 + \frac{x^{12}}{12} + \frac{1}{x^8}$

g) $h(x) = \frac{4x^2+3x^3+21}{x^2}$

3. Find the value of;-

a) $f'(1)$

b) $f'(2)$ in each of the following functions

i. $f(x) = \frac{x^2-9}{x+3}$

ii. $f(x) = \frac{x^3+2x-4x^4}{x}$

iii. $f(x) = \frac{x^2-9}{x+3} + \frac{x^4-16}{x-2}$

Derivative of composite functions

A composite function is a function that consists of more than one function in it. If a function $f(x)$ is a composite function of function g of h and g and h are functions of x the function

$$f(x) = g(h(x))$$

A composite function may have two or more functions for instance a function $e(x)$ may be made of a function f of g of h all which are functions of x. Such a function is expressed as

$$e(x) = f(g(h(x)) \quad or$$

To evaluate $e(x)$, x is substituted in $h(x)$, the output is substituted $g(x)$, and that output is substituted in $f(x)$.

For instance in the expression,

$$y = \sqrt{(x^2)^3}$$

The function $y = f(x)$ is a result of several functions such as

$$h(x) = x^2$$

$g(x) = g(h(x))$, which is the cube of $h(x)$,

$$g(x) = (h(x))^3 = (x^2)^3 \quad and$$

$f(x)$ which is the square root of $g(h(x))$.

$f(x) = \sqrt{g(x)} = \sqrt{(x^2)^3}$ hence there are three functions:

$$h(x) = x^2$$

$g(x) = x^3$

$f(x) = \sqrt{x}$

The chain rule

The chain rule states that $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dv} \cdot \frac{dv}{dx}$ when we a composite of three functions.

If the composite function has two functions, $\frac{dy}{dx} = \frac{dy}{dv} . \frac{dv}{dx}$ Where $y = f(x) = v(u(x))$

Example 1

Find the value of $f'(x)$ if $f(x) = (3x^2 + 4)^7$

Solution

Let $y = f(x) = u^7$

where $u = 3x^2 + 4$

$\frac{dy}{du} = 7u^6$, $\frac{du}{dx} = 6x$

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$= 7u^6. 6x$

$= 7(3x^2 + 4)^6. 6x$

$= 42x (3x^2 + 4)^6$

Example 2

Find the value of $f'(x)$ if $f(x) = \sqrt{x^7 + 1}$

Solution

Let $y = f(x) = \sqrt{u}$ where $u = x^7 + 1$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} \qquad \frac{du}{dx} = 7x^6$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times 7x^6$$

$$= \frac{1}{2}(x^7 + 1)^{-\frac{1}{2}} \times 7x^6 = \frac{7}{2}x^6(x^7 + 1)^{-\frac{1}{2}} = \frac{7x^6}{2\sqrt{x^7+1}}$$

$$f'(x) = \frac{7x^6}{2\sqrt{x^7+1}}$$

Example 3

Find the value of $f'(2)$ given that $f(x)$

$f(x) = (x^5 - 2)^8$

Solution

$y = (x^5 - 2)^8$

Let, $y = u^8$, where $u = x^5 - 2$

$$\frac{dy}{du} = 8u^7$$

$$\frac{du}{dx} = 5x^4$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 8u^7.5x^4 = 8(x^5 - 2)^7 \times 5(x^4)$$

$$\frac{dy}{dx} = 40x^4(x^5 - 2)^7$$

$$f'(2) = 40 \times 2^4(2^5 - 2)^7 = 1.4 \times 10^{13}$$

Exercise 3.4: To be done in groups of four students

1) Find the derivative.

   a) $y = (x^3 + 2)^3$

   b) $f(x) = (x^3)^8$

   c) $f(x) = \sqrt{x^4}$

   d) $f(x) = \sqrt{x^3 + 2x + 1}$

   e) $f(x) = [\sqrt{x^2 + 3x + 1)^3}]^7$

   f) $f(x) = \sqrt[7]{\left(\frac{x^4 - 16}{x^2 + 4}\right)^3}$

   g) $f(x) = \sqrt[8]{\frac{20x^2 + 9x + 1}{4x + 1}}$

   h) $f(x) = (3x^2 + 3)^7$

   i) $f(x) = \left(\frac{3}{x^4} + x^4 + x^3 + x^2\right)^2$

2) A curve has a function $y = (x^3 + 7x + 3)^3$ find the gradient of the curve at points

   i. (0, 27)

   ii. (-1, -125)

3) Find the value of

   i. $f'(0)$

   ii. $f'(2)$

   For the following functions;

   a) $f(x) = (x^3 + 1)^4$

b) $f(x) = \sqrt{x^3 + \frac{1}{x^2} + 2}$ for $x \leq 0$

Tangent and Normal to the Tangent of a Curve

The tangent line

A tangent of a curve is a straight line that touches the curve at only one point. For a function $f(x)$, each of point (x) has its tangent. The gradient function of the function $f(x)$ is the gradient of the tangent at the given value of x. A point where a tangent touches a curve is called a point of tangency.

Figure 3.3 below shows the graph of a function $f(x) = x^2$. The tangent to the curve at point (0, 0) is illustrated with an emboldened line. The gradient of the tangent at any point is the same as that of the function.

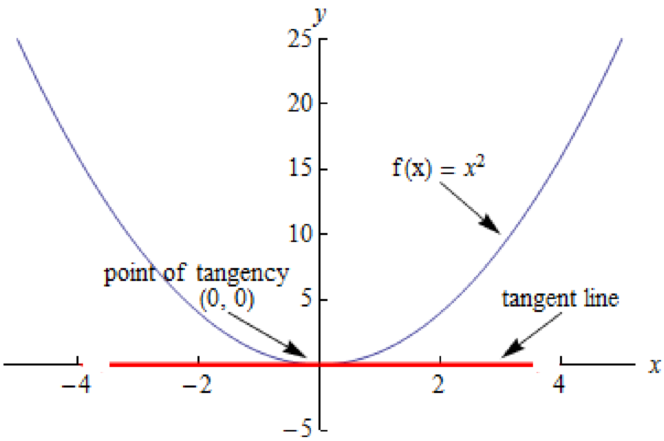


Fig 3.3 Gradient of tangent = gradient function (derivative) of the function

The curve is $f(x) = x^2$

The gradient function $f'(x) = 2x$

Substituting for x at the point of tangency to $f'(x)$ we obtain the gradient of the tangent. $f'(0) = 0$

On substituting the gradient, point of tangency and general point (x, y) we obtain the equation of tangent.

$$\frac{y-0}{x-0} = 0$$

$$\frac{y}{x} = 0$$

$$y = 0$$

Hence the equation of the tangent is $y = 0$ (the x-axis)

Example 1

Determine the equation of the tangent to the function
$f(x) = x^3 + 3x + 4$ at a point (0, 4)

Solution

$$f(x) = x^3 + 3x + 4$$

$$f'(x) = 3x^2 + 3$$

At (0, 4) $f'(0) = 3.0^2 + 3$

$f'(0) = 3$, the gradient at point of tangency is 3

The tangent passes through (x, y), (0, 4) and has gradient 3. The equation is:

$$\frac{y-4}{x-0} = 3$$

$$\frac{y-4}{x} = 3$$

$$y - 4 = 3x$$

$$y = 3x + 4$$

The Normal Line

The normal line is a straight line that is perpendicular to the tangent at the point of tangency.

Example 1

Study figure 3.5 below and answer the questions that follow. In the figure is the curve of a function y = f(x) = x [2] , i ts tangent and the normal at point (1, 1) .

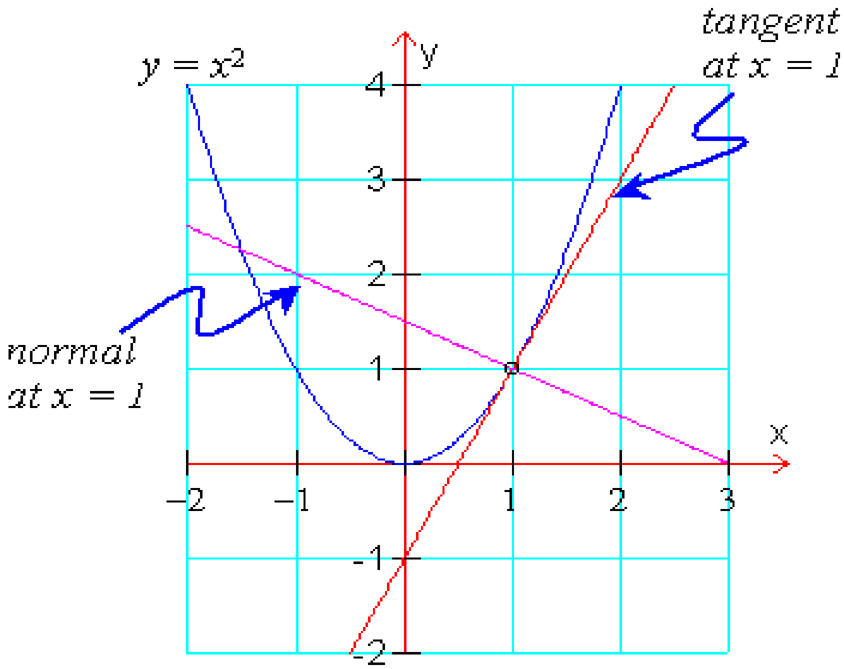


Fig 3.4

a. What is the relationship between the gradient of the tangent and the normal at the point of tangency ?
b. Determine the equation of the normal line at (1, 1).

Solution

From Secondary 2, the product of the tangent and its normal is equal to -1 since they are perpendicular lines. Using the tangency point and a general point. It is possible to determine the equation of the normal line. The equation of the normal shown in the figure above is determined as shown below.

$y = f(x) = x^2$ is equation of the line.

$\frac{dy}{dx} = f'(x) = 2x$ is the gradient function of the curve.

At point (1, 1)

Gradient of tangent $M_1 = f'(1) = 2 \times 1 = 2$

Gradient of normal, M_2 is calculated using $M_1 M_2 = -1$ since they are perpendicular.

$2M_2 = -1, M_2 = -\frac{1}{2}$

The normal has gradient $M_2 = -\frac{1}{2}$ and pass through (1, 1) and (x, y).

Its equation is hence

$$\frac{(y-1)}{(x-1)} = -\frac{1}{2}$$

$$2(y-1) = -1(x-1)$$

$$2y - 2 = -x + 1$$

$$2y = -x + 3$$

$$y = -\frac{1}{2}x + \frac{3}{2}$$

Task

Show that at point (1, 8) on the curve $f(x) = x^3 + 2x^{2+} + 3x + 2$

The tangent has equation $y = 10x - 2$, and the normal has equation

$$y = -\frac{1}{10}x + \frac{81}{10}$$

Exercise 3.5: Work in groups of four students.

1. Given the equation of a curve is a function in such that

$y = f(x) = x^3 - x + 10$ find the equation of

i. The equation of the tangent at (1, 10)

ii. The equation of the normal at (1, 10)

2. A function $f(x)$ is described such that for $f(x), f'(5) = 2$ and $f(5) = -3$ at $x = 5$. Determine
   i. The equation of the tangent at the given point.
   ii. The equation of the normal at the given point.

3. Find the equation of the tangent of the following functions at the stated point.
   a) $y = \sqrt{12x}$ at $(3, 6)$
   b) $f(x) = x^2 - 8x$ at $(3, -15)$
   c) $f(x) = 2x^2 - 3x - 2$ at $(0, -2)$
   d) $f(x) = (x^2 + 2x + 1)^{12}$ at $(0, 1)$

4. Determine the equation of the normal to the following functions at the stated point.
   a) $f(x) = x^3 + 2x$ at (0, 0)
   b) $f(x) = \frac{x^4 - 16}{x + 4}$ at $x = 0$
   c) $f(x) = x^3$ at (3 , 27)
   d) $f(x) = \sqrt{x}$ at (4, 2)
   e) $f(x) = (x^2 + 2)^8$ at (0 , 256)
5. Figure 3.5 below shows the movement made by a ball hit by a football player. Study it and answer the questions that follow.

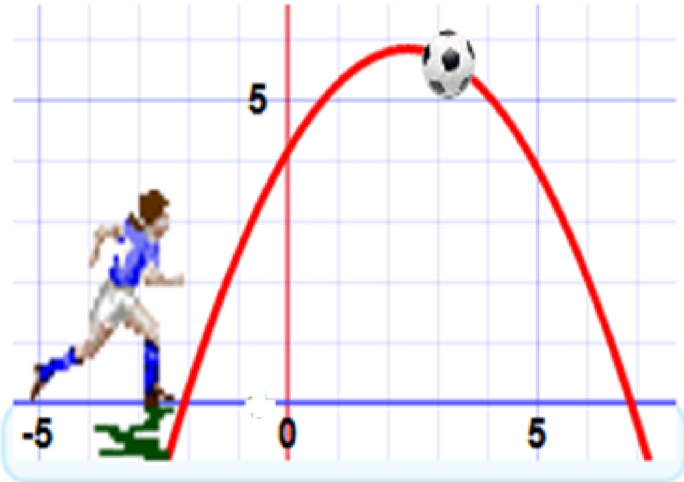


Figure 3.5

a) Determine the equation of the curve.

b) Calculate $f'(x)$ at x = 1.

c) Calculate the normal to the curve at x = 5.

Second Derivative

Definitions

The derivative of a function $f(x) = f'(x)$ is called first derivative of the function $f(x)$. The notation for the first derivative $of\ y = f(x)$ are

$$\frac{dy}{dx} \text{ or } \frac{df(x)}{dx} \text{ or } f'(x)$$

If the first derivative of a function f', is a differentiable function at a value x then its derivative is called the second derivative of the function $f(x)$. The second derivative is the rate of change of the gradient function.

The notation for second derivative

The notation for second derivative are $\frac{d^2y}{dx^2}$ or $\frac{d^2f(x)}{dx^2}$ or $f''(x)$ or $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

The first derivative, $\frac{dy}{dx}$, describes how a variable y changes with respect to a variable x at all real values of x with a defined y.

The second derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ describes how the first derivative changes with respect to x.

Recall, if $y = ax^n$ $then$ $\frac{dy}{dx} = anx^{n-1}$

Second Derivative Power Rule

The first derivative of a general function $y = ax^n$ is $\frac{dy}{dx} = anx^{n-1}$

The second derivative of the general function is $\frac{d^2y}{dx^2} = a.n.(n-1)x^{n-2}$

Example 1

Given the function $f(x) = 3x^4$ find

a) f'
b) f''

Solution

i. $f(x) = 3x^4$

$f'(x) = 4 \times 3x^{4-1}$

$f'(x) = 12x^3$

ii. $f''(x) = 12x^3$
$f''(x) = 3 \times 12x^2$

$f''(x) = 36x^2$

Exercise 3.6: Work in pairs.

1. Find the second derivative of the following functions

   a) $f(x) = 4x^{12}$

   b) $f(x) = \frac{12}{13}x^{26} + x^4$

   c) $f(x) = x^4 - 3x^2 + 2x + 4$

   d) $f(x) = 4x^3 + 8x^3 - 6x^2 + 2x + 14$

2. Find the value of f'' for the following.

   a) $f(x) = 2x + x^3$

   b) $f'(x) = 20x^2 + \frac{1}{5}x^{10}$

   c) $f'(x) = 24x^{\frac{1}{6}} + 12x^6$

   d) $y = 2x^3 + 3x^2 - 16$

3. Find the values of $f''(3)$ in the function

   $f(x) = 3x^4 - 2x^2 + 4x + 3$

4. If $\frac{dy}{dx} = 24x^7 + 7x^6$ find the value of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ at the point where x =1.

5. A quantity s is a function of t represented by the equation $s = t^3 + 2t$. Determine an expression that explains how the rate of change of s with respect to t changes with respect to t.

Application of differential functions

The stationary points of a function

Stationary points of a function $y = f(x)$ are points at which there is instantaneously no change in the slope of the curve. At these points the tangent to the curve is horizontal and the gradient of $f(x)$ is zero.

At stationary point of a function $f(x)$, the derivative $f'(x) = 0$.

Stationary points include maximum, minimum and point of inflexion.

Stationary points are also known as critical points of functions. Figure 3.7 below shows curves with marked maximum, minimum and point of inflexion.

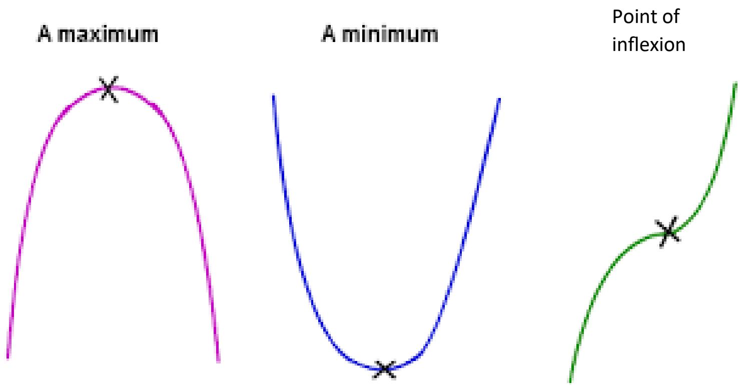


Fig 3.6

Methods of identifying stationary points include

1. Using the first derivative
2. Using the second derivative.

Using the first derivative to identify stationary points

The Maximum Point of a Curve

A maximum point of a function $f(x)$ at $x = c$ is a point at which

$f'(c) = 0$

x	$x < c$	$f(c)$	$x > c$
f'	$f' > 0$	$f' = 0$	$f' < 0$
$f(x)$	increasing	stationary	decreasing

Figure 3.8 below shows a sketch in change in gradient at values of x close to a maximum stationary point.

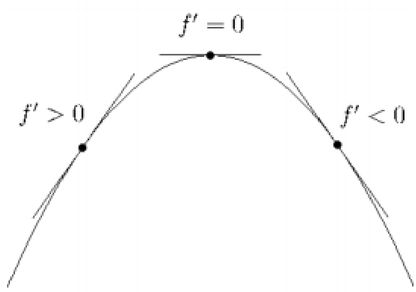


Fig 3.7 Maximum

At a maximum point the curve has $f' = 0$.

The Minimum Point of a Curve

A minimum point of a function $f(x)$ at $x = c$ is a point at which

$f'(c) = 0$,

x	$x < c$	$f(c)$	$x > c$
f'	$f' < 0$	$f' = 0$	$f^1 < 0$
$f(x)$	decreasing	stationary	increasing

Figure 3.9 below shows a sketch of change in gradient at values of x close to a minimum stationary point.

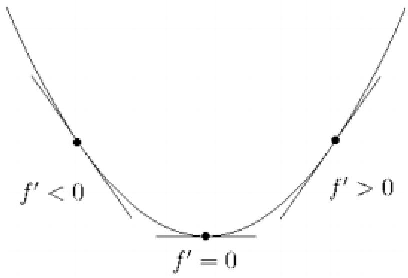


Fig 3.8 Minimum

The Point of Inflexion of a Curve

A point of inflexion of $f(x)$ is a point $(c, f(c))$ where $f'(c) = 0$.

Unlike the minimum or maximum, the gradient will either continue positive if the graph is increasing (e.g. $y = x^3$ at $(0,0)$) or negative if the graph is decreasing (e.g. $y = 1 - x^3$ at $(0,1)$).

The number of Stationary Points

If a function $f(x)$ is a polynomial where the greatest power of x is n, the function has at most $(n-1)$ stationary points. Such a polynomial is called a polynomial of degree n, A polynomial of degree n has at most $n-1$ turning points.

Task

You are given a function y= f(x) = $x^2 - 6x + 1$. Determine the coordinates of the stationary point.

a) Complete the table.

x	$x < c$	$x = c$	$x > c$
f'		0	

$f(x)$			

b) State the type of stationary point.

Task

In pairs, show that the function $y = f(x) = x^3 - 3x^2 + 8$ has turning points at (0, 8) and (2, 4) and determine their type.

Example 3

Determine the stationary points of the function $f(x) = x^3 + 5$ and state their type.

Solution

$f(x) = x^3 + 5$

$f'(x) = 3x^2$

$f'(x) = 0$ at stationary point hence $f' = 3x^2 = 0, x = 0$ (repeated roots)

$f(0) = 0 + 5 = 5$, the stationary point is (0 , 5)

x	$x = -1$	$x = 0$	$x = 1$	$x = 2$
$f'(x)$	$f'(-1) = 3$	0	$f'(1) = 3$	$f'(2) = 12$
$f(x)$	increasing	stationary	increasing	increasing

The stationary point (0, 5) is a point of inflexion.

Exercise 3.7a

In each of the following functions, determine the stationary point(s) and their nature using the first derivative.

a) $y = x^3 + 3x^2 - 2$

b) $y = 2x^3 - 4$

c) $f(x) = -2(x-3)^2 + 8$

d) $f(x) = -x^3 - x$

e) $f(x) = 2x^3 + 3x^2 + 5$

1) Using the Second Derivative to determine the nature of a stationary point

Task

Sketch graphs of each type of stationary point (minimum, maximum and point of inflexion). Make a note on graph whether the function is increasing or decreasing (i.e. the gradient function positive or negative) on either side of the stationary point.

a) What can be said about the rate of change of the gradient function in each case?

b) Test out your answer on the functions in exercise 3.7a.

Exercise 3.7b

1. For each of the following functions

i. Determine the stationary point.

ii. Determine the nature of the stationary point using the second derivative.

a) $f(x) = -x^3$

b) $f(x) = -x^3 - x + 4$

c) $f(x) = 2x^3 + 3x^2 + 5$

d) $f(x) = -8(x-3)^3 + 27$

e) $f(x) = 2 - 3x - x^3$

f) $f(x) = 12x^3 - 6x^2 + 9x + 1$

Displacement, Velocity and Acceleration

The study of displacement, velocity and acceleration and their relationship is called **kinematics.**

Displacement is the distance covered in a given direction. Its symbol is s.

Velocity (v) is rate of change of displacement with respect to time (t).

Velocity is the derivative of displacement.

$v = \frac{ds}{dt}$

Acceleration (a) is the rate of change of velocity with respect to time. The differential function of velocity is the acceleration. When the acceleration is zero, the velocity is not changing so it must be a constant.

In general $s = f(t)$, called displacement.

$v = f'(t) = \frac{ds}{dt}$, called velocity.

$a = f''(t) = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{dv}{dt}$, called acceleration.

Task

The distance moved by a plane from point A is expressed by the function $s = 3t^4 + 6t^2 + 30$ in metres after time t in seconds. In groups of three students find

a) The distance of the plane from point A before it started moving.
b) The distance moved by the plane in the first 10 seconds.
c) Calculate the velocity of the plane at $t = 5$ seconds.
d) The acceleration of the plane at $t = 2$ seconds.

Example 3.8: Work in pairs.

1. A particle moves on a straight line according to the expression $s(t) = 2t^3 - 6t^2 + 3$. At what time is the particle moving at a constant velocity?
2. A stone is thrown up and the height travelled by the stone is $h(t) = -5t^2 + 5t + 100$ Calculate
   a) The height at 3 seconds.
   b) The velocity at 2 seconds.
   c) The acceleration at 5 seconds.
   d) At what time was the velocity maximum?
3. Find the acceleration at $t = 2$ seconds for a displacement function $s(t) = 3t^3 + t^2 + t + 4$.
4. At what time is acceleration of a particle moving according to the function $s(t) = 5t^3 - 24t^2 + 44$ equal to 6 m/s².
5. A plane moves according to the function $s(t) = 3 - 16t + 4t^2$. Find the acceleration of the plane at $t = 22$ sec.
6. If $s(t) = 3t^3 + 2t^2 + 4t + 10 = 0$, find;
   a) $s(0)$
   b) $s(2)$
   c) $v(1)$
   d) $a(2)$

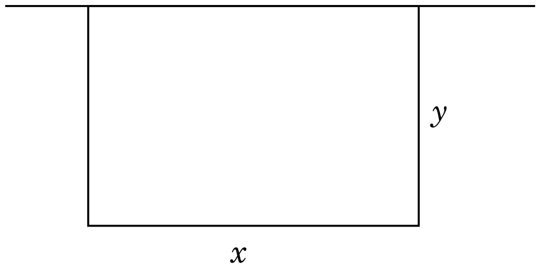
Optimization

If the relationship between two variables can be expressed as a function it is possible to optimize one quantity in terms of the other. We optimise to find the greatest (maximum) or least (minimum) value of a function.

Example

A 500m mesh wire is to enclose three sides of a rectangular field against a wall. Determine the dimensions that will enclose the largest possible area.

Solution



Perimeter = $x + 2y$

Area = xy

To maximize Area (A) against dimension find A' (dimension)

From $x + 2y = 500$, $y = \frac{500-x}{2}$

$$A = xy = \frac{x(500-x)}{2}$$

At maximum area $\frac{dA}{dx} = 0$

$\frac{dA}{dx} = 250 - x = 0$ when $x = 250$

But $x + 2y = 500$

$2y = 250$

$y = 125$

Maximum area, has length 250 m and width 125 m, 31 250 m^2.

Exercise 3.9: Work in groups. Present your findings to the class.

1. Determine the dimensions that will minimize the cost to build a closed box with a volume of 0.9 m^3. Given that the length of the box is three times its width and the material used to make the

side faces costs SSP 60 per m^2 and that of base and top is SSP 100 per m^2.

2. Determine the dimensions of a cylinder that is made using the minimum of metal plate to give a volume of $1500cm^3$ ($1.5dm^3$).
3. A farmer wished to fence a rectangular field next to a concrete boundary wall. He decided to fence on the three sides away from the wall at a cost of SSP 20 per metre. If he is to fence 18 hectares of land.

   Calculate:

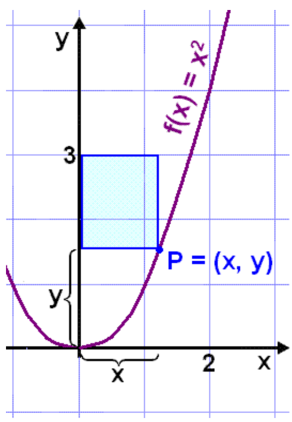
   a) The dimensions of the field that require the minimum of fencing.

   b) The minimum cost of fencing.
4. Jane wishes to make an open cuboid with a surface area of $1080cm^2$ and a square base.

   a) Calculate the maximum volume of the box made.

   b) Calculate the cost of half filling the box with milk that cost SSP 50 per litre.
5. Find the maximum volume of an open cylinder made from a metallic plate $10m^2$.

6. A rectangle is drawn as shown in the diagram. One side is formed by the line $y = 3$ and one corner, P, lies on the graph of $f(x) = x^2$. Find the coordinates of the point P in order for the rectangle to have the maximum possible area.



UNIT 4: CALCULUS 2 (INTEGRATION)

Introduction

As discussed in calculus 1, when a function *f(x)* is known and is differentiable, its derivative is expressed as $\frac{d\,f(x)}{dx}$ or $\frac{dy}{dx}$ or $f'(x)$. When the differential function is known, the process of determining the function $y = f(x)$ is called **integration.** Integration is the inverse of differentiation.

For instance the differential function of $y = x^3$ is $\frac{dy}{dx} = 3x^2$

Differentiation

$$x^3 \rightleftarrows 3x^2$$

Integration

To **differentiate** x^n multiply by the power of *x* and decrease the power of *x* by 1. The inverse, which is **integration** involves increasing the power of *x* by 1 and dividing by the new power of *x*.

When differentiating $y = ax^n, \frac{dy}{dx} = anx^{n-1}$.

When integrating ax^n we get $\int ax^n dx = \frac{ax^{(n+1)}}{(n+1)}$

Task

Copy and complete table 4.1 below on derivatives of functions

Table 4.1

Function of *x*	Derivative $f'(x)$
$f(x) = x^3$	
$f(x) = x^3+3$	

$f(x) = x^3 + 5$ $f(x) = x^3+8$ $f(x) = 9$	

a. What do you notice about the *f(x)* and *f'(x)*?

A constant has zero gradient, consequently when integrating there are many possible functions, so a constant c (called a constant of integration) is added to the expression obtained.

$$y = \int ax^n \, dx = \frac{ax^{(n+1)}}{n+1} + c$$

The value of constant c can be determined when more details are given. Integral that require a constant of integration are known as **indefinite integrals**. The function obtained from integration, without *c*) is called the **integrand.**

Exercise 4.1: Working in pairs.

1. Evaluate

a. $\int 2x \, dx$
b. $\int 3x^2 dx$
c. $\int 7x^2 dx$
d. $\int x^8 dx$
e. $\int \frac{1}{2} x^4 dx$
f. $\int x^{\frac{1}{2}} dx$
g. $\int x^6 dx$

2. Find the indefinite integral with respect to *x* of the following expressions

a) $2x^2$
b) x^3
c) x^4
d. $x^{\frac{1}{5}}$
e. $x^{\frac{1}{6}}$
f. $4x^8$

Example 1

Evaluate $\int 6x + 3x^4 dx$

Solution

$$\int 6x dx + \int 3x^4 dx$$

$$= 6\int x dx + 3\int x^4 dx$$

$$= \frac{6x^2}{2} + \frac{3x^5}{5} + c$$

Note: only one constant of integration is necessary.

Exercise 4.2: Working in pairs.

1. Find the integral of the following expressions against x.
   a. $\sqrt{2x}$
   b. $(x^2 + 7)^2$
   c. $(5x^2)^3$
   d. $\frac{3x-9}{x^8}$
   e. $\frac{x^2-4}{x+2}$
   f. $\frac{x^4+5x+6}{(x+3)}$

2. Find the value of
   a. $\int x^2 + 11x - 25\, dx$
   b. $\int t^4 - t^3 + t^2 + t - 1\, dt$
   c. $\int 6x^8 - 20x^4 + x^2 + 9\, dx$
   d. $\int x^6 - 4x^4 + x\, dx$
   e. $\int 40t^3 - 12t^2 + 9t + 14\, dt$
   f. $\int 40t^3 - 12t^2 + 9t\, dt + 13$
   g. $\int \frac{x^2+4}{x+2} + \frac{x^2-4}{x-2}\, dx$
3. Determine $f(x)$ given that $f'(x) = x^7 + 6x^2$
4. A curve has gradient function $\frac{dy}{dx} = 3x + 4$ and passes through a point (2, 16). Determine the equation of the curve.

Integration by substitution

Substitution is used to simplify the integration of composite functions. Recall: composite functions are differentiated by use of chain rule. The chain rule states

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Reversing this process we get the composite function such that,

$$\int f(x)dx = \int h(x) \times h'(x)\, dx$$

This method is used to find an integral of a composite function that is a result of the chain rule in which the integrand function can be written in term of two functions, where one function is the derivative of the other.

Integrating powers of algebraic functions

Example 1

Evaluate $\int (x+2)^3\, dx$

Solution

By substitution let $u = x + 2$

$$\frac{du}{dx} = 1,\ du = dx$$

Substituting u and du in the expression,

$$\int \mathrm{u}^3\, du = \frac{1}{4}\mathrm{u}^4 + \mathrm{c}$$

Substituting back $u = x + 2$

$$\int (\mathrm{x}+2)^3 dx = \frac{1}{4}(x+2)^4 + c$$

Integrating products of algebraic functions

Example 2

Find the value of the following indefinite integral.

$\int x^3\sqrt{x^4+1}\,dx$

Solution

Let $u = x^4 + 1$

$\frac{du}{dx} = 4x^3$

$dx = \frac{du}{4x^3}$

Substituting u and dx

$\int x^3 u^{\frac{1}{2}} \frac{du}{4x^3} = \frac{1}{4}\int u^{\frac{1}{2}}\, du$

$= \frac{1}{4} \times \frac{2}{3} u^{\frac{3}{2}} + c$

$= \frac{1}{6} u^{\frac{3}{2}} + c$

Substituting u back.

$\int x^3\sqrt{x^4+1}\,dx = \frac{1}{6}(x^4 + 1)^{\frac{3}{2}} + c$

Note: Whilst c is used in both the integrals (with respect to u and with respect to x), it is unlikely they will be equal.

Integration of functions with fractions

Example 3

Evaluate $\int \frac{4x}{\sqrt{x^2+2}}\, dx$

Solution

Let $u = x^2+2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

Substituting for dx and u in the intgral

$\int \frac{4x}{u^{1/2}} \cdot \frac{du}{2x} \quad = 2\int u^{-1/2}$ du

$= \frac{2u^{\frac{1}{2}}}{\frac{1}{2}}$ +c

$= 4u^{\frac{1}{2}} + c$

Substituting dx and u.

$\int \frac{4x}{\sqrt{x^2+2}}\, dx = 4(x^2+2)^{\frac{1}{2}} + c = 4\sqrt{x^2+2} + c$

Exercise 4.3: Work in groups.

1. Find the value of the indefinite integrals
   a) $\int (x+1)^2\, dx$
   b) $\int \sqrt{7t+9}\, dt$
   c) $\int (x^2 + 7)^2 dx$
   d) $\int (5t^2)^2\, dt$
   e) $\int (5x+11)^4 dx$
   f) $\int (3-x)^{10} dx$

2. Find the value of indefinite integral of;
   a) $\int 12(3x+1)^3$dx
   b) $\int (x^2+5x)^7(2x+5)\, dx$
   c) $\int (2x+3)(x^2+3x)^2$dx
   d) $\int 2x\sqrt{2x^2}+1\, dx$
   e) $\int \frac{x}{\sqrt{x^2+1}}$ dx

3. Find the value of indefinite integral of:
   a) $\int \frac{8x^2}{(x^3+1)^2}\, dx$
   d. $\int \frac{x}{(x^2+1)^2}\, dx$

b) $\int \frac{8x}{(1-x^2)^4}\,dx$

e. $\int \frac{x^3}{(x^4+1)^{\frac{1}{3}}}\,dx$

c) $\int \frac{(3x-9)}{x^3}\,dx$

4. A function *f(x)* has the gradient function $\frac{dy}{dx} = \frac{3x}{(1+9x^2)^3}$. What is the value of f*(x)*?

Application of integration

In calculus 1, application of differentiation in kinematics was discussed. Since integration is the inverse of differentiation, the same applications apply.

Example

The velocity v m/s of a particle that travels on a straight line at time t from a fixed point Q is $v = 3t^2 - 18t - 32$ where $t \geq 0$, find

a. Value of t when $a = 0$ m/s²
b. The distance of the particle from Q at minimum velocity.

Solution

a. $v = 3t^2 - 18t + 32$

$\frac{dv}{dt} = a = 6t - 18$

at $a = 0$, $6t - 18 = 0$, and $t = 3\ s$

b. $v = 3t^2 - 18t + 32$

$s = \int v\, dt = \int (3t^2 - 18t \mp 32)dt$

$s = t^3 - 9t^2 + 32t + c$, at $t = 0$, $s = 0$ hence $c = 0$

$s = t^3 - 9t^2 + 32t$

At minimum velocity, $a = \frac{dv}{dt} = 0$ hence $t = 3\ s$

$s(3) = 3^3 - 9 \times 3^2 + 32 \times 3 = 42$ m

Task 1

In groups, find out what the derivatives represent given that s =displacement, v =velocity and a =acceleration.

a. $\frac{ds}{dt}$
b. $\frac{dv}{dt}$
c. $\frac{d^2s}{dt^2}$

Task 2

In groups, find out what the integrals represent given that s =displacement, v =velocity and a =acceleration.

a. $\int a\, dt$
b. $\int v\, dt$

Exercise 4.4

To be done in groups

1. **A c**ar moves in a straight line at a speed of 20 m/s. At point A the driver noticed an accident at point B and applied a deceleration of $a = \frac{3t}{2} - 6$.
   After four seconds the car passed point B. It then accelerated at a = 2 m/s^2 to attain the original speed of 20m/s by point C. Find
   a. The speed of car at B.
   b. The distance AB
   c. The time taken for the car to travel from B to C.
2. A motorist left point A and travels at a velocity $v = 6t - \frac{1}{2}t^2$ to rest at point B. If t is time in minutes from leaving point A, find,
   a. The time taken to move from A to B
   b. The distance AB.
   c. The acceleration at t = 8 min.

UNIT 5: MATRICES

Introduction

A matrix is a rectangular arrangement of numbers. As learnt earlier matrices can be formed from simultaneous equations of word problems. Consider the following relationship

The cost of a pen and a book is SSP 3

The cost of 3 pens and 1 book is SSP 5

Using the cost of a pen as p and the cost of a book as b we can form simultaneous equations

$p + b = 3.$

$3p + b = 5.$

From this equation we can form three types of matrixes namely:

Coefficient matrix from coefficient ordered p and b, which is $\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$.

The variable matrix from the variable p and b which is p and b which is $\begin{pmatrix} p \\ b \end{pmatrix}$.

The constant matrix from the solutions of equations, which is $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

As learnt previously you should be able to recall the order of a matrix, types of matrices, inverse matrices, determinants and solutions of problems using 2×2 matrices. In this unit we will mainly discuss 3×3 matrices.

Using scientific calculator to operate on matrices

The Scientific calculator and technology provides quick and accurate method for solving matrix problems. Investigate how to use your calculator's matrix functions.

Exercise 5.1: Work in pairs.

1. Write the order of the following matrices

   (a) $\begin{pmatrix} 2 & 2 \\ 4 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   (c) $\begin{pmatrix} 4 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

   (d) $\begin{pmatrix} 8 & 9 \\ 4 & 2 \\ 3 & 4 \end{pmatrix}$

2. Simplify the matrix products below using a calculator or otherwise.

   (a) $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 4 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

3. Given $A = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ find the value of:

   a) $2A$

   b) A^2

   c) $2A + 3B$

   d) $2A + B^2$

4. Find the determinant of the following matrices

a) $\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 4 & 7 \\ 2 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 4 & 2 \\ 0 & 1 \end{pmatrix}$

5. Find the inverse of the following matrices

a) $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ d) $\begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$ e) $\begin{pmatrix} 7 & 2 \\ 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$ f) $\begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix}$

6. a) What is a singular matrix?

b) Solve for *x* if the following matrices are singular matrices.

i. $\begin{pmatrix} 2x & 1 \\ 8 & 4 \end{pmatrix}$

ii. $\begin{pmatrix} x & 4 \\ 8 & 2x \end{pmatrix}$

iii. $\begin{pmatrix} x & 3 \\ -2 & x+1 \end{pmatrix}$.

7. A triangle with vertices A (2, 7) , B(2, 3) and C(5 ,3) was transformed to $\Delta A_1B_1C_1$ using a transformation matrix of $\begin{pmatrix} 2 & 4 \\ -1 & 4 \end{pmatrix}$

Find the area of $\Delta A_1B_1C_1$.

8. Use matrices to solve the following pairs of simultaneous equations.

a) $p + b = 3$
$3p + b = 5$

b) $2a + 3b = 13$
$a + 2b = 8$

c) $2x + y = 31$
$x + 3y = 43$

d) $4x + y = -12$
$2x + 7y = 24$

Determinant of a 3 × 3 matrix

There are many ways to obtain the determinant of a 3 × 3 matrix. Two are included here:

1. Using a scientific calculator
2. Using cofactors and minors

Using a scientific calculator

Task

Use a calculator to show the determinant of the matrix.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ -4 & 0 & 1 \\ 3 & 0 & -1 \end{pmatrix}, \det A = -1$$

Exercise 5.2: In pairs.

Determine the determinant of the following matrices using a scientific calculator.

1. $\begin{pmatrix} -1 & -2 & -2 \\ -2 & 1 & 1 \\ 3 & 4 & 5 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 4 & 5 \\ -1 & -2 & -2 \\ 2 & 1 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{pmatrix}$

6. $\begin{pmatrix} 8 & -1 & 9 \\ 3 & 1 & 8 \\ 11 & 0 & 17 \end{pmatrix}$

Cofactor expansion

Cofactor expansion is a method used to find the determinant of a matrix by use of the matrix of cofactors and the matrix itself.

This method can only be discussed after discussing:

a) A minor and the minor matrix
b) A cofactor and the cofactor matrix

A minor and the matrix of minors

As discussed earlier, the position of any given element of a matrix can be described using subscripts *e.g.* an element a in row *i* and column *j* of a matrix is written as a_{ij}

1. Using the matrix: A = $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{pmatrix}$

$a_{11} = 2$, $a_{12} = 3$, $a_{13} = 4$

$a_{21} = 3$, $a_{22} = 2$, $a_{23} = 1$

$a_{31} = 4$, $a_{32} = 3$, $a_{33} = 2$

The symbol for a minor is M. A minor is an element M_{ij} obtained by determining the determinant of a matrix left on deleting the *i*–row and *j*-column of the matrix.

Example 1

Determine the minors for matrix A = $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{pmatrix}$

Solution.

The minors for this matrix are:

$$M_{11} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$M_{12} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6 - 4 = 2$$

$$M_{13} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} = 9 - 8 = 1$$

$$M_{21} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 3 & 2 \end{vmatrix} = 6 - 12 = -6$$

$$M_{22} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} = 4 - 16 = -12$$

$$M_{23} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6 - 12 = -6$$

$$M_{31} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 3 - 8 = -5$$

$$M_{32} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 2 - 12 = -10$$

$$M_{33} = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5$$

The number of minors of a matrix is the same as the number of elements of the matrix. The matrix formed by the minors is called the minor matrix. Its symbol is M. The matrix of minors for matrix *A* is

$$M = \begin{pmatrix} 1 & 2 & 1 \\ -6 & -12 & -6 \\ -5 & -10 & -5 \end{pmatrix}$$

A cofactor and the matrix of cofactors (C)

A cofactor of a matrix is an element C_{ij} that is derived from the expression

$$C_{ij} = (-1)^{(i+j)} M_{ij}$$

A cofactor is a signed minor. The sign of the cofactor is (-1) to a power of $(i + j)$ where i is the row and j is the column of the matrix.

Since the power (-1) alternates between -1 and + 1 with even and odd powers respectively since the sum of *i* and *j* alternate even and odd the sign for the cofactor matrix is

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$ This is multiplied by the matrix of minors (*M*) to give the matrix of cofactors (C).

Example 2

State the matrix of cofactors for the matrix, $A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{pmatrix}$

Solution

$$C = \begin{pmatrix} +1 & -2 & +1 \\ +6 & -12 & +6 \\ -5 & +10 & -5 \end{pmatrix}$$

The product and sum of the elements in a particular row or column of the original matrix and the matrix of cofactors, gives the determinant of the matrix.

Note: it does not matter the row or column you choose for instance a_{11} for matrix and c_{11} multiplied together implies use of the 1st row of the matrix multiplied with the 1st row of the cofactors that are multiplied to give the determinant.

$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$

Given $A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -2 & 1 \\ 6 & -12 & 6 \\ -5 & 10 & -5 \end{pmatrix}$,

$\det(A) = 2 \times 1 + -2 \times 3 + 4 \times 1 = 0$ 1st row

or

$\det(A) = 3 \times 6 - 12 \times 2 + 6 \times 1 = 0$ 2nd row

or any other row or column

Since C is derived from the signed minor matrix (M) and for the first row the signs are $(+,-,+)$ and the minors are the determinants of 2 × 2 matrices it is possible to define the determinant of a 3 × 3matrix as

$$\det(A) = |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Task

In groups of four show that the determinant of matrix $A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$ is 165.

Exercise 5.3: Work in pairs.

Find the determinant of the following matrices using cofactor expansion

1. $\begin{pmatrix} 8 & -1 & 9 \\ 3 & 1 & 8 \\ 11 & 0 & 17 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 3 & 49 \\ 3 & 2 & 1 \\ 4 & 3 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 4 \\ 0 & 6 & 0 \end{pmatrix}$

6. $\begin{pmatrix} 3 & -2 & 4 \\ 2 & -4 & 5 \\ 1 & 8 & 2 \end{pmatrix}$

7. $\begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$

Note

The method of cofactor expansion can be used to find the determinant of other square matrices such as 4 × 4, 5 × 5, …$n \times n$ order matrices

A scientific calculator can also be used to calculate the determinant of matrices.

Inverse of a 3 × 3 and other matrices

The inverse of matrix A is written as A^{-1}

It has the property

$AA^{-1} = A^{-1}A = I$

Where I is the identity matrix, a square matrix whose elements are zero except for those on the leading diagonal that are all 1.

The 3 × 3 identity matrix is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrices cannot be divided, so multiplication by the inverse allows us to find e.g. B in A.B = C

Multiplying both sides by A^{-1}

A^{-1} A B = A^{-1} C

Since A^{-1}A= I we get

I B = A^{-1}C

Since I is identity: I B = B and hence B = A^{-1} C

Inverse of a 2 × 2 matrix

As learnt earlier the inverse of a 2 × 2 matrix is obtained by pre-multiplying a matrix by the reciprocal of its determinant and interchanging elements in leading diagonals and changing the sign of the elements in the secondary diagonal.

For instance;

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Where det(A) = ad – bc

If the determinant is zero, $\frac{1}{\det A}$ will be undefined and the matrix will not have an inverse. A matrix without an inverse is called a Singular Matrix.

Task 1

Show that the inverse of matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Inverse of a 3×3

Inverse of a 3×3 and bigger matrices is determined via more tedious methods. There are two main methods of determining these inverse matrices. They include:-

1. Gauss Jordan Elimination/row reduction
2. Using a scientific calculator.

Using Gauss Jordan Elimination / Row Reduction

In this method, to find the inverse of A, A^{-1}, a larger matrix of A and the identity matrix I is formed. (A | I)

Using available elements of A one manipulates rows across the two matrixes in order to reduce A to I, at the same time I is converted to A^{-1} . The example below illustrates the reduction process.

Example

Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

Solution

Write (A | I) and convert A to I and I to A^{-1} by operations on the rows.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 1 & 1 \end{array}\right)$$

Subtracting row 1 from row 3

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

Subtracting 2×row 1 from row 2

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

Adding 2×row 2 to row 3, and multiplying row 3 by -1

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Adding 3×row 3 to row 2

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Subtracting 2×row2 from row 1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -15 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

We have reduced (A| I) to (I | A^{-1}) the inverse. Therefore

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Using a calculator to determine inverse of a matrix

Task

In pairs, use a calculator to check the previous result.

Exercise 5.4: Work in pairs

Calculate the inverse of the matrix below using Gauss Jordan Elimination

1. $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}$

3. $\begin{pmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{pmatrix}$

4. $\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 3 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix}$

Use a calculator to determine the inverse of the following matrices

6. $\begin{pmatrix} 2 & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$

8. $\begin{pmatrix} -1 & 1 & 4 \\ -6 & 3 & 4 \\ 2 & 0 & 6 \end{pmatrix}$

9. $\begin{pmatrix} -3 & 4 & 1 \\ -2 & 6 & -4 \\ -5 & 4 & 5 \end{pmatrix}$

Solving simultaneous linear equations using the inverse of a matrix

The knowledge of inverse matrices enable us to solve problems involving matrix multiplication. Using the inverse of a 3 × 3 matrix enables us to solve systems of linear equations with three unknowns by forming a matrix equation.

A matrix equation is an equation comprising a **coefficient matrix, variable matrix** and **constant matrix.**

The coefficient matrix is a 3 × 3 matrix formed from the coefficients of variables arranged in the same order, A

The variable matrix is a 3 × 1 (column) matrix that contains the variables, B

The constant matrix is a 3 × 1 column matrix that contain the constants, C

A.B = C

If A.B = C then $A^{-1}A.B = A^{-1}C$

If we multiply both sides of the equation by the inverse of A then the RHS simplifies to $B = A^{-1}C$

From $B = A^{-1}C$ it is possible to obtain the solution to the simultaneous equations.

Task

Solve the simultaneous equation.

$x + 2y - z = 7$

$2x - 3y - 4z = -3$

$x + y + z = 0$

Using calculator to solve the simultaneous equations

It is possible to obtain the solutions of simultaneous equations from the calculator directly.

Task

Attempt in groups.

A quadratic equation of a curve has a general form of $y = ax^2 + bx + c$ and passes through points (-2, 4), (2, 2) and (4, 4). Determine the equation of the curve.

Hint: substitute for *x* and *y* for each of the points to derive three simultaneous equations for *a, b* and *c.*

Exercise 5.5: Solve in groups. Present your findings to the class.

1. Use matrices to solve the simultaneous equations below.

a) $2x + 3y + y = 5$
$x + y + z = 3$
$x + 2y + 2z = 6$

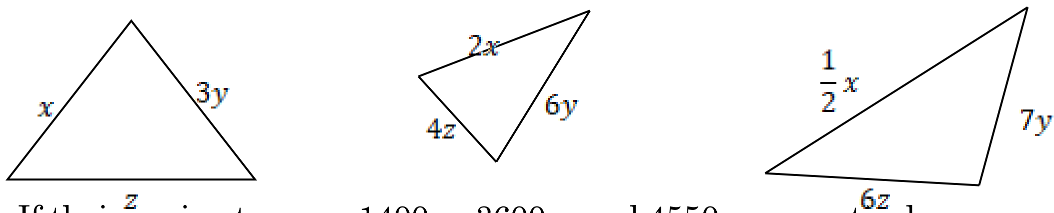
b) $x + 3y + z = 15$
$2x + y + z = 10$
$4x + 7y + 3z = 40$

c) $x + z = 6$
$3x + 4y + z = 18$
$6zx + y + 3z = 31$

d) $x + 2z + 3y = 12$
$4x + 3z + 2y = 28$
$6x + 2x + y = 50$

e) $z + 2x + y = 5$
$x + 3x + 4z = 12$
$2x + 2z + 7y = 6$

2. In South Sudan the cost of a pen, a book and a ruler is SSP 45. The cost of 2 books, 4 rulers and a pen is SSP 110 and 7 books, 6 rulers and 2 pens is SSP 250. Find the cost of each item.
3. The figures below shows three triangular piece of land with marked dimensions.



If their perimeters are 1400m, 3600m and 4550m respectively. Determine the values of x, y and z.

UNIT 6: COMPLEX NUMBERS

Introduction

The squares of positive numbers and negative numbers give a positive number, for instance

$(-3)^2 = 9.$

$(+3)^2 = 9.$

$(4)^2 = 16.$

$(4)^2 = 16.$

It is possible to write that

$(\pm 3)^2 = 9.$

$(\pm 4)^2 = 16.$

Since the square root of a number is obtained by inversing the square we notice that

$\sqrt{9} = \pm 3 = 3 \text{ or} -3.$

$\sqrt{16} = \pm 4 = 4 \text{ or} -4.$

In this scenario we do not encounter a negative square since the square of any real number is always a positive number. Similarly, there is no square root of a negative number. However, some problems may give the square root of a negative number.

For instance in solving a quadratic equation $x^2 + 3x + 4 = 0$ by completing the square $\left(x + \frac{3}{2}\right)^2 + 4 - \frac{9}{4} = 0$

$$\left(x + \frac{3}{2}\right)^2 + \frac{7}{4} = 0$$

$$\left(x+\frac{3}{2}\right)^2=-\frac{7}{4}$$

$$x+\frac{3}{2}=\pm\sqrt{-\frac{7}{4}}=\pm\frac{\sqrt{-7}}{2}$$

$$x=-\frac{3}{2}\pm\frac{\sqrt{-7}}{2}$$

You notice that in our previous units we say x has no real roots since it has not been possible to find square root of negative numbers such as – 7.

It is important to note that: $\sqrt{-9}\neq 3$, $\sqrt{-1}\neq 1$. In this unit, we will discuss how to determine such roots.

To simplify such expressions we introduce the small letter i such that $\sqrt{-1}=i$.

Squaring both sides we get $i^2=-1$.

Real numbers, imaginary and complex numbers.

Complex numbers are numbers that consist of two parts — a real number and an imaginary number. Complex numbers are the building blocks of more intricate mathematics, such as algebra. They can be applied to many aspects of real life, especially in electronics and electromagnetism.

The standard format for complex numbers is $a + ib$, with the real number first and the imaginary number last. Because either part could be 0, technically any real number or imaginary number can be considered a complex number. Complex does not mean complicated, it means that the two types of numbers combine to form a complex, like a housing complex — a group of buildings joined together. It is hence a set that contains subsets of real numbers and imaginary numbers.

Real numbers are tangible values that can be plotted on a horizontal number line. Imaginary numbers are abstract concepts that are used when you need the square root of a negative number.

The symbol for complex number is small letter 'z' hence.

$z = a + ib$.

Where: z is complex number

a and b are real numbers

i is the imaginary number $\sqrt{-1}$.

Graphical Representation of complex numbers

Graphs of complex numbers are called **Argand Diagrams**. These graphs are representations of the **complex plane**. A complex plane is similar to the Cartesian plane. It has a horizontal axis that represent the real part of the complex number a vertical axis that represents the imaginary part of the complex number.

When graphing the complex number $z = x + iy$ we locate z at a point (x, y).

Figure 6.1 below shows the Argand diagram with a general complex numbers $z = x + yi$.

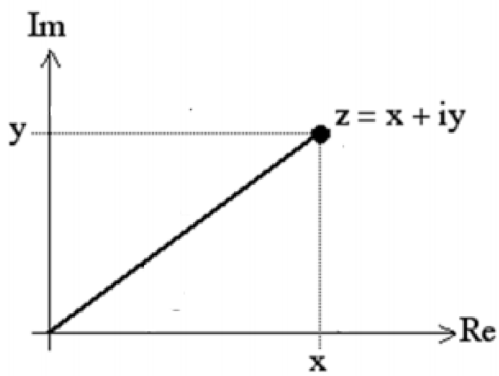


Figure 6.1

Example 1

Draw the graph of the complex numbers $z = 6 + 4i$

Solution

Plot (6, 4).

Figure 6.2 below shows $z = 6 + 4i$

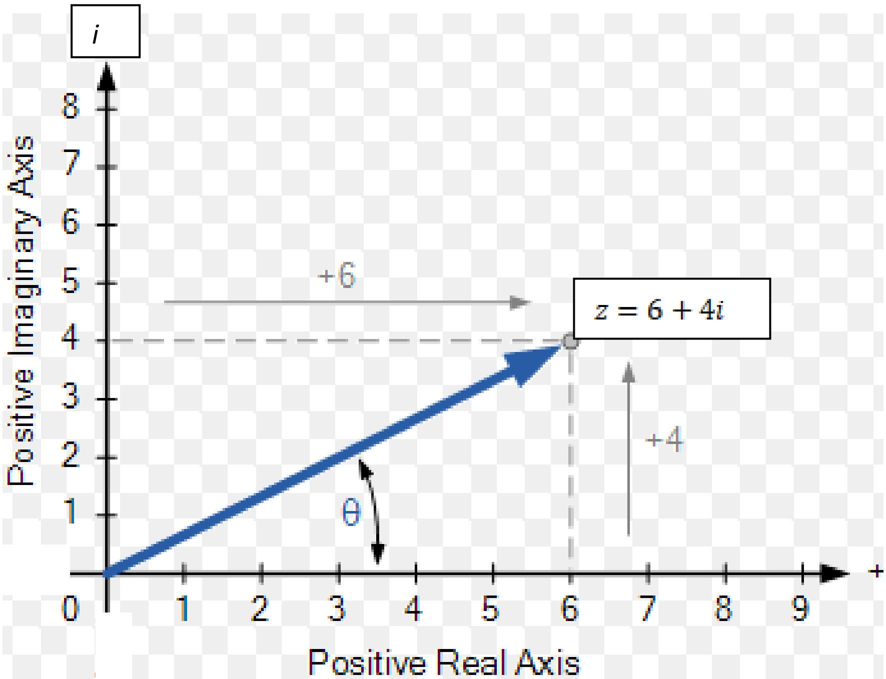


Figure 6.2

Task

Draw a Cartesian plane and show the position of the following complex numbers.

i) $z_1 = 4 + 3i$

ii) $z_2 = 0 + 4i$

iii) $z_3 = -2 + 3i$

iv) $z_4 = -4$

v) $z_5 = 3 - 2i$

Exercise 6.1

Work in groups.

1. Figure 6.3 below shows position of complex numbers z_1, z_2, z_3, z_4 and z_5 starting from positive real axis anticlockwise. Study it and answer the questions that follow.

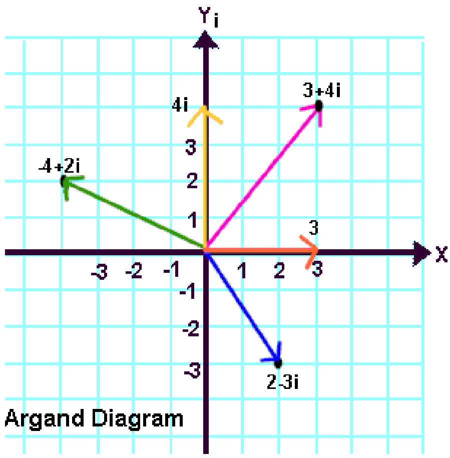


Figure 6. 3

Determine the complex numbers $z_1 \rightarrow z_5$

2. Study figure 6.4 below and write the values of the complex numbers z_1, z_2, z_3, z_4 and z_5

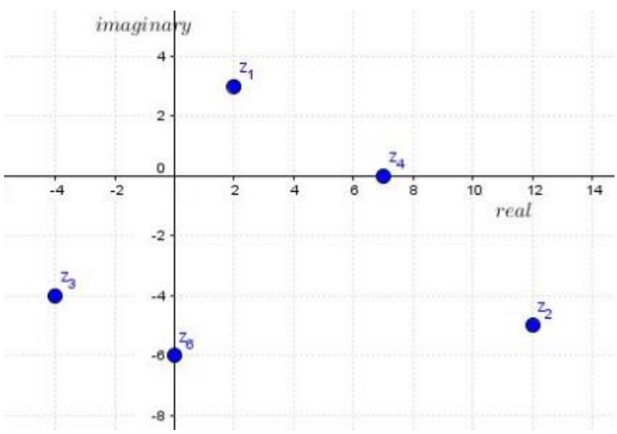


Figure 6.4

Simplifying imaginary numbers

Imaginary numbers are simplified by introducing i.

Example 1

Simplify $\sqrt{-16}$

Solution

Introduce $\sqrt{-1} = i$

$\sqrt{-16} = \sqrt{16 \times -1}$.

$= \pm\sqrt{16} . \sqrt{-1}$.

$= \pm 4i$.

Example 2

Simplify $\sqrt{\frac{-25}{81}}$

Solution

$$\sqrt{\frac{-25}{81}} = \frac{\sqrt{-25}}{\sqrt{81}} = \frac{\sqrt{25 \times (-1)}}{\sqrt{81}}$$

$$= \frac{\sqrt{25} . \sqrt{-1}}{\sqrt{81}}$$

$$= \frac{\pm 5i}{9}$$

Exercise 6.2: Work in pairs.

Simplify the following numbers by writing them in terms of imaginary numbers.

1. $\sqrt{9}$
2. $\sqrt{-4}$

3. $\sqrt{\frac{-9}{25}}$

4. $\sqrt{\frac{-25}{47}}$

5. $\sqrt{-4}$

6. $\sqrt{-121}$

7. $\sqrt{441}$

8. $\sqrt{(-4)\times(-16)}$

9. $\sqrt{-169}$

10. $\sqrt{-2}$

11. $\sqrt{-3}$

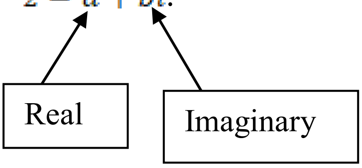
12. $\sqrt{-100}$

13. $\sqrt{-81}$

Addition and subtraction of complex numbers.

A complex numbers has a real and an imaginary part.

$z = a + bi.$



To add or simplify two or more complex numbers we simplify the different parts differently for instance

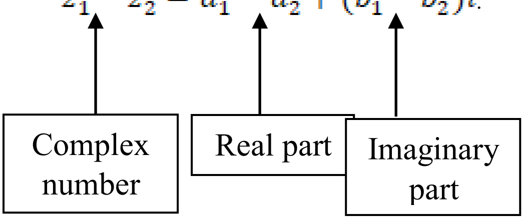
$z_1 = a_1 + b_1 i.$

$z_2 = a_2 + b_2 i.$

$z_1 + z_2 = a_1 + a_2 + b_1 i + b_2 i.$

$z_1 + z_2 = a_1 + a_2 + (b_1 + b_2)i.$

$z_1 - z_2 = a_1 - a_2 + (b_1 - b_2)i.$



Task

Given the complex numbers, $z_1 = 2 + 3i$ and $z_2 = 3 + i$. In pairs

i. Find the value of ,

a) $2z_1$
b) $3z_2$
c) $z_3 = z_1 + z_2$
d) $5z_1 + 7z_2$

i. Draw an Argand diagram of z_1, z_2, and z_3

ii. What do you notice about the addition of complex numbers?

The sum or difference of complex numbers can be illustrated in the same way as addition or subtraction of vectors.

Exercise 6.3: Work in pairs.

1. Add or subtract the following complex numbers
   a) $3 + 2i + 4$
   b) $4i + 3i + 7$
   c) $4 - 2i + 3i$
   d) $4 - 9 - i - 7i$
2. Simplify the following
   a) $3(2 + i)$
   b) $-3(-2 + 2i)$
   c) $\frac{1}{2}(-3 + 4i)$
   d) $3(2 + 4i) - 4(3 + 2i)$
   e) $\frac{1}{3}(2 - 5i) - \frac{1}{6}(12 + 2i)$
3. Given that $z_1 = 3i + 4$

   $z_2 = 6i + \frac{1}{2}.$

   $find\ the\ value\ of.$
   a) $3z_1$
   b) $3z_2 - Z_1$
   c) $3z_1 - 2z_2$
   d) $z_1 - 12z_2$

e) $\frac{1}{2}z_1 + \frac{1}{7}z_2$

Multiplying complex numbers

A complex number has two terms. It therefore looks like a binomial expression. When two complex numbers are multiplied the solution is similar to that of an expansion.

Task

Copy and complete the table below

n	0	1	2	3	4	5	6	7	8
i^n	i^0	i^1	i^2	i^3					
simplified	1		-1						

Plot the points on an Argand diagram. What do you notice?

Task

In pairs, expand the following expressions.

1. $(a+bi)(a+bi)$
2. $(a+bi)(a-bi)$
3. $(a-bi)(a+bi)$

   What do you notice?

Example

Simplify the following complex numbers

a) $(3+2i)(4+2i)$

b) $(3+2i)^2$

Solution

a) $(3+2i)(4+2i)$

$= 12 + 6i + 8i + 4(i)^2$

$= 12 + 14i - 4$

$= 8 + 14i$

But $(i)^2 = -1$

b) $(3+2i)^2 = (3+2i)(3+2i)$

$= 3(3+2i) + 2i(3+2i)$

$= 9 + 6i + 6i + 4(i)^2$

$= 9 + 12i + 4(-1)$

$= 5 + 12i$

But $(i)^2 = -1$

Task

You are given that $z_{1=} (3+2i)$ and $z_{2=} 4+2i$, in pairs ,

a. Find the value of $z_3 = z_1' = i\,(3+2i)$ and $z_4 = z_2' = i\,(4+2i)$
b. Illustrate $z_{1,}z_2, z_1'$ and z_2' on an Argand diagram.
c. What transformation is represented by multiplication by i?

Exercise 6.4: Solve in groups.

1. Simplify the following complex numbers by expansion.
   a) $(3+i)(3+2i)$
   b) $(3-2i)(4-2i)$
   c) $(2i+2)(3i-1)$
   d) $(8-3i)(2-5i)$
2. Find the value of the following expression.
   a) $\sqrt{(4+3i)(4-3i)}$
   b) $\sqrt{(25-5i)(25+5i)}$
   c) $\sqrt{\left(\frac{1}{4}-\frac{1}{3}i\right)\left(\frac{1}{3}i+\frac{1}{4}\right)}$
3. If $z = 2+3i$ find
   a) z^2
   b) z^3
   c) z^4

Conjugate of complex numbers and complex conjugation

Task

At the start of the chapter we found the solutions to the equation

$x^2 + 3x + 4 = 0$ are $x = -\frac{3}{2} \pm \frac{\sqrt{-7}}{2}$.

Write these solutions as complex numbers and plot them on an Argand diagram. What do you notice?

Add them together. What do you notice?

Multiply them together. What do you notice?

The solutions of quadratic equations with real coefficients always come in pairs $a + ib$ and $a - ib$. They are known as conjugate pairs, and their sum and product are both real.

The term conjugate was introduced in previous unit when discussing conjugate of surds. If a surd is multiplied by its conjugate the surd is eliminated from a problem.

For instance

$(\sqrt{3} + \sqrt{2})$ has a conjugate $(\sqrt{3} - \sqrt{2})$

And $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 9 - \sqrt{6} + \sqrt{6} - 2 = 9 - 2 = 7$

Similarly, the conjugate of a complex number is a complex number such that the product of the complex number and its conjugate gives a real number.

The symbol for conjugate of a complex number z is z^*

The modulus of a complex number

In complex numbers, the modulus is the length of a line segment from the origin to the point on the Argand diagram.

The symbol for modulus of a complex number is $|z|$ where $|z| = \sqrt{z . z^*}$

Task 1

Given the complex numbers $z_1 = 5 + 2i$, $z_2 = 3 + 4i$.

In pairs, find the value of

i. $z_1 + z_2$
ii. z_1^*
iii. z_2^*
iv. $z_1 z_2^*$
v. $|z_1|$

Exercise 6.5: Work in pairs.

1. Given that $z = 3 + 2i$ find the value of $z + z^*$
2. Determine the modulus of
   a) $3 - 2i$
   b) $3i - 1$
   c) $4 + 3i$
   d) $2 + i$
3. If $z_1 = 2 + 3i$ and $z_2^* = 3 + 2i$
   Find the value of
   a) z_1^*
   b) z_2
   c) $z_1 z_2$
   d) $|z_1 + z_2|$
   e) $|3z_1 + 2z_2|$

Dividing complex numbers

If a number has a complex denominator it should be simplified to have a non-complex number or non-imaginary number in the denominator. This can be done by multiplying both the numerator and denominator of such a fraction by the conjugate of the denominator.

In general.

If $z_3 = \frac{z_1}{z_2}$

z_3 can be simplified as shown below.

$z_3 = \frac{z_1}{z_2} \times \frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|}.$

Example 1

Simplify

a) $\frac{1}{2i}$

b) $\frac{2+3i}{4-3i}$

Solution

a) $\frac{1}{2i} = \frac{1}{2i} \times \frac{-2i}{-2i} = \frac{2i}{-4(i)^2} = \frac{-2i}{4} = -\frac{1}{2}i.$

b) $\frac{(2+3i)}{(4-3i)} = \frac{(2+3i)(4+3i)}{(4-3i)(4+3i)} = \frac{8+6i+12i+9(i)^2}{16+12i-12i-9(i)^2}$

$= \frac{8+18i-9}{16+9} = \frac{-1+18i}{25} = \frac{1}{25}(-1+18i).$

Exercise 6.6: Solve in groups.

1. Simplify the following complex number fractions.
   a) $\frac{3+2i}{3+i}$
   b) $\frac{1+3i}{2+i}$
   c) $\frac{3+4i}{2i}$
   d) $\frac{i}{2+i}$
2. Simplify

a) $\frac{12+6i}{1+2i}+\frac{2+4i}{i}$
b) $\frac{3+2i}{1+3i}+\frac{4+5i}{1-3i}$
c) $\frac{2+4i}{3+4i}+\frac{3+3i}{2+5i}$

3. Given that $z_1 = 2+4i, z_2 = 3+2i$ and $z_3 = 7+5i$
   Find.
   a) z_1^*
   b) $|z_1+z_3|$
   c) $z_1+3z_2+4z_3$
   d) $(z_2+z_3)^*$
   e) $\frac{z_1}{z_2}$
   f) $\frac{z_1}{2z_2+z_3}$

Solving Quadratic Equations with Complex Numbers

A quadratic equation is of the form $ax^2+bx+c=0$. The general formula for solving a quadratic equation is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The value $\sqrt{b^2-4ac}$ is called the discriminant and it typifies the solutions for the equation. If

1. $b^2-4ac=0$ The equation has repeated (i.e. two equal) roots.
2. $b^2-4ac>0$ The equation has two distinct real roots.
3. $b^2-4ac<0$ The equation has two complex roots.

Extending the number system to complex numbers means all quadratic equations have two roots.

Example 1

In pairs, solve the quadratic equations below.

a) $x^2 - 4 = 0$

b) $x^2 + 9 = 0$

Solution

a) $x^2 - 4 = 0$

$x^2 = 4$

$x = \pm\sqrt{4} = \pm 2$

b) $x^2 + 9 = 0$

$x^2 = -9$

$x = \pm\sqrt{-9} = \pm\sqrt{-9}\ \sqrt{-1} = \pm 3i$

Example 2

Find the solutions of $10x^2 + 6x + 1 = 0$

Solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{36 - 40}}{20}$$

$$x = \frac{-6 \pm \sqrt{-4}}{20}$$

$$x = \frac{-6 \pm 2i}{20}$$

$$x = \frac{-6}{20} \pm \frac{2i}{20}$$

$$x = \frac{-3}{10} \pm \frac{2i}{10}$$

$x = \frac{-3}{10} + \frac{2i}{10}$ or $x = \frac{-3}{10} - \frac{2i}{10}$

Example 3

Two numbers are such that, their sum is four and their product is 10. If one of the numbers *is x.*

i. Write an expression for the other number

ii. Determine the numbers

Solution.

$4 - x$ is the other number

$x(4 - x) = 10$

$4x - x^2 = 10$

$x^2 - 4x + 10 = 0$

Using the quadratic formula

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times (10)}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{16 - 40)}}{2 \times 1} = 2 \pm \frac{\sqrt{-24}}{2} = 2 \pm \frac{\sqrt{-6 \times 4}}{2}$$

$$= 2 \pm \frac{2\sqrt{-6}}{2} = 2 \pm \sqrt{-6} = 2 \pm \sqrt{6}\, i$$

Exercise 6.7

1. Solve the following quadratic equations and state the types of roots.

a) $3x^2 + 4x + 5 = 0$
b) $14x^2 + 9x + 10 = 0$

2. A farmer intends to make a flower garden as shown in figure below of the area $75m^2$. The ratio of the sides are 3:1, what dimensions will his garden be?

UNIT 7: THE CIRCLE

Introduction

A circle is the locus of all points equidistant from a fixed point called the centre. The common distance between these points from the centre is called the radius of the circle. When a circle is drawn on a Cartesian plane many of its properties and points can be described. For such a circle it is possible to determine, the coordinates of the centre of the circle, coordinates of points at its circumference, the equation of the circle and the equation of the tangent to the circle at any point on the circumference.

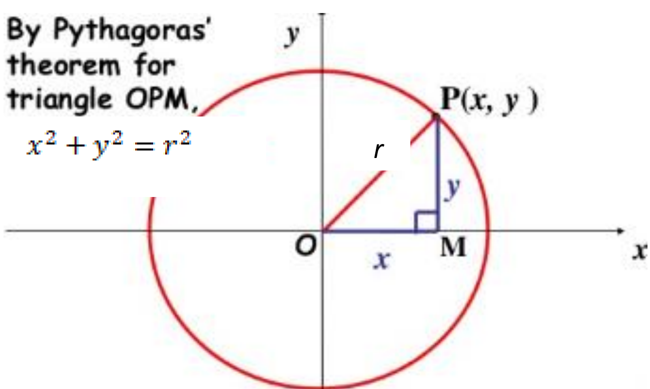
The equation of a circle

There are two main scenarios in describing the equation of a circle. They include when the centre of the circle is the origin (0, 0) and when the centre is not the origin (i.e. a general point (h, k) where h and k are real numbers not equal to zero).

The equation of a circle, centre the origin

Example 1

Figure 7.1 shows a circle centre the origin (0, 0) passing through a general point P(x, y). ΔOMP is a right angled triangle with sides OM = x and PM = y and OP = radius..



By Pythagoras' theorem $x^2 + y^2 = r^2$

This is the general equation of a circle, centre the origin (0, 0)

Example 2

Find the equation of a circle centre (0, 0) with radius $\sqrt{17}$ *units*

Solution

$x^2 + y^2 = r^2$.

Given $r = \sqrt{17}$, $r^2 = (\sqrt{17})^2 = 17$.

$x^2 + y^2 = 17$ is the equation of the circle.

Example 3

Find the centre and radius of the circle whose equation is
$2x^2 + 2y^2 = 16$

Solution

Simplifying $2x^2 + 2y^2 = 16$

$$x^2 + y^2 = 8$$

Comparing with $x^2 + y^2 = r^2$

$r^2 = 8$

$$r = \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$$

The circle has centre (0, 0) and radius $r = 2\sqrt{2}$

Exercise 7.1: Solve in groups of three. Present your findings to the class.

1. Find the equation of a circle with centre the origin and the radius given in (a - f) below.

a) $r = 3$

b) $r = 5$

c) $r = \sqrt{13}$

d) $r = \frac{1}{2}$

e) $r = 1\frac{1}{2}$

2. Find the radius of the circles whose equations are given below.

   a) $x^2 + y^2 = 25$

   b) $x^2 + y^2 = 100$

   c) $x^2 + y^2 = 81$

   d) $x^2 + y^2 = 1$

   e) $x^2 + y^2 = 29$

   f) $x^2 + y^2 = 21$

   g) $9x^2 + 9x^2 = 25$

3. Determine the equation of each circle whose centre is the origin and that passes through the given point below.

   a) $(3,4)$

   b) $(0,3)$

   c) $(-2,0)$

   d) $(-1,1)$

   e) $(-1,-5)$

   f) $(0,12)$

   g) $(\frac{1}{2},\frac{1}{3})$

   h) $(3\frac{1}{2},4\frac{1}{2})$

4. Calculate the area of the circles

   a) $x^2 + y^2 = 49$

   b) $x^2 + y^2 = 9$

   c) $x^2 + y^2 = 16$

   d) $x^2 + y^2 = 2$

   e) $x^2 + y^2 = 7$

   f) $x^2 + y^2 = 36$

The equation of a circle centre

Example 1

Study the *figure 7.2* below and answer the questions that follow.

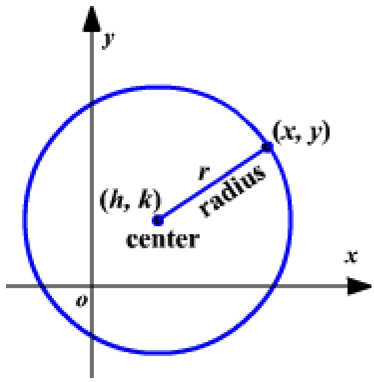
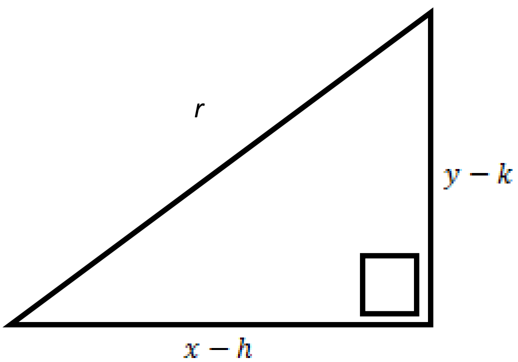


Figure 7.2

a. What do you observe about the diagram?
b. Using the distance formula, express the square of the radius in terms of x, y, h and k.

Solution

You notice the figure has a circle whose centre is point (h, k) it passes through a general point (x, y) at the circumference and has radius r.



In the right angled triangle

$r^2 = (x - h)^2 + (y - k)^2$

This expression is called the general equation of a circle centre (h, k) and radius r units.

Example 2

A circle has an equation $(x - 2)^2 + (y + 3)^2 = 25$. Find the radius and the centre of the circle.

Solution

Compare the equation with general equation hence

$(x-2)^2+(y+3)^2=25$

$(x-h)^2+(y-k)^2=r^2$

You notice

$r^2 = 25$ and hence

r = 5 units

$(x-2)^2=(x-h)^2$ hence $x-2=x-h$ and simplifying $h=2$

$(y+3)^2=(y-k)^2$.

$y+3=y-k$.

$k = -3$.

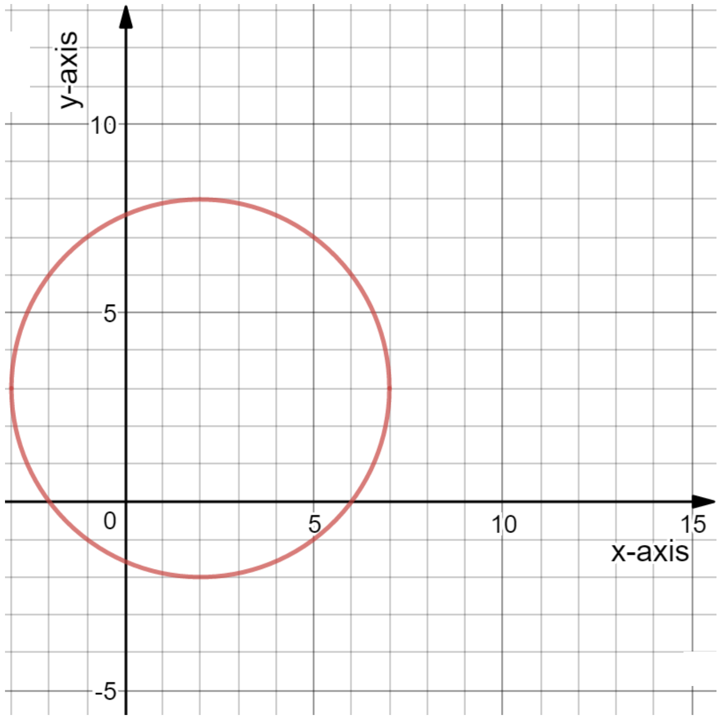
Hence $h = 2$ *and* $k = -3$ hence so centre (h,k)*is* $(2,-3)$

The circle has centre $(2,-3)$ and radius 5 units.

Example 3

Determine the equation of a circle centre $(2,3)$ with a point $P(5,7)$ at circumference.

Solution



Notice the relationship shown.

To have solutions for $(x-h)^2+(y-k)^2=r^2$, h = 2 and k = 3

Radius $r=\sqrt{(7-3)^2+(5-2)^2}$

$r=\sqrt{16+9}=\sqrt{25}=5.$

Inserting $r^2 = 25$, h = 2, and k = 3 in general equation we obtain

$(x-2)^2+(y-3)^2=25.$

Exercise 7.2: Work in pairs.

1. Use a graphing program like Desmos or Geogebra. Find the centre and the radius of a circle whose equations are
   a) $(x-2)^2+(y+5)^2=9$
   b) $(x+3)^2+(y-5)^2=4$
   c) $(x+a)^2+(y+b)^2=c^2$

d) $(x-2)^2+(y+5)^2=121$
e) $(x+2)^2+(y-7)^2=441$
f) $(x-\frac{1}{2})^2+(y+\frac{1}{2})^2=0.25$
g) $(y-2)^2+(x-3)^2=3$
h) $(y+17)^2+(x+2)^2=17$

2. Find the equations of a circle given the coordinate of the centre and its radius in (i) – (iii) below.
   i) Centre (1 , 2), radius = 2
   ii) Centre (-1 , 4), radius = $\frac{1}{2}$
   iii) Centre (-4 , -3), radius = 3
   iv) Centre (8 , 4), radius = 16
   v) Centre (0.3 , 0.7), radius = 0.25
   vi) Centre $(\frac{-1}{3},\frac{-1}{3})$, radius = $\frac{1}{3}$
   vii) Centre (7 , 8), radius = 14
3. Find the equation of a circle given its centre and one of its point at circumference in the following questions:
   a) Centre (0 , 0) and pass through (6 , 8)
   b) Centre (2 , 3) and pass through (8 , 11)
   c) Centre (-4 , -3) and pass through (-1 , 1)
   d) Centre (-12 , -7) and pass through (4 , 5)
   e) Centre (7 , 2) and pass through (16 , 14)
   f) Centre $(-3\frac{1}{2},-4\frac{1}{2})$ and pass through $(-2,-2\frac{1}{2})$

Find the equation of a circle given two points on its diameter

Example 1

a. How does the diameter of a circle relate to the radius?
b. If points A (x_1,y_1) and $B(x_2,y_2)$ are at intersection of the circumference and diameter of a circle. How do these points relate to the circle?

Solution

A diameter forms a straight-line equivalent to two radius of a circle.

The middle most point of extreme ends of a diameter is the centre of the circle.

Using the relationship that the midpoint of a point A (x_1, y_1) and $B(x_2, y_2)$ is

$M = \left(\frac{(x_1+x_2)}{2}, \frac{(y_1+y_2)}{2}\right)$. This point represent the centre of the circle as shown in figure 7.3

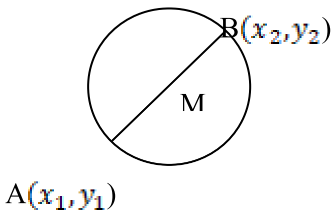


Figure 7.3

The coordinates of M are the centre of such a circle.

The radius is half the length AB.

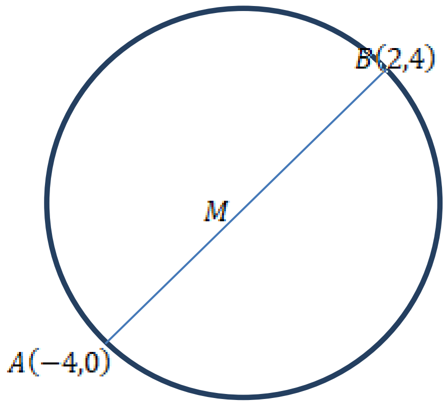
The equation of a circle can hence be obtained by substituting the centre's coordinate and the square of radius into the general equation.

$(x-h)^2 + (y-k)^2 = r^2$

Example 2

The end points of the diameter are $A(-4,0) and\ B(2,4)$. in groups of three, find the equation of this circle.

Solution



The coordinate of M (the mid-point) is $\left(\frac{2-4}{2}+\frac{0+4}{2}\right)$

$M=\left(\frac{-2}{2},\frac{4}{2}\right)=(-1,2)$ is the centre of the circle.

The diameter squared, $(2r)^2=(y_2-y_1)^2+(x_2-x_1)^2$

$=(y_2-y_1)^2+(x_2-x_1)^2$

$=(4-0)^2+(2--4)^2$

$$=4^2+6^2=16+36=52$$

$r^2=\frac{52}{4}=13$

$r=\sqrt{13}$

Substituting centre and radius to general equation

$(x-h)^2+(y-k)^2=r^2.$

$(x--1)^2+(y-2)^2=13.$

$(x+1)^2+(y-2)^2=13.$

Exercise 7.3: Work in pairs.

Find the equation of a circle given the two points below are opposite co-ordinates of the diameter of the circle.

1. $(-4,3)$ and $(2,1)$
2. $(2\ ,3)$ and $(5,7)$
3. $(-2,4)$ and $(4,11)$
4. $(-4,-7)$ and $(8,9)$
5. $(5,7)$ and $(\frac{13}{2},9)$
6. $(-21,-18)$ and $(-9,-2)$

The expanded general equations of a circle

Example

In pairs,

a. Expand $(x-h)^2+(y-k)^2=r^2$ and arrage the terms in order of powers of the variables
b. What do you notice about the coefficient of x^2 and y^2
c. Hence, express r in terms of h, k and c.

Solution

$(x-h)(x-h)+(y-k)(y-k)=r^2$

$x^2-xh-hx+h^2+y^2-2yk+k^2=r^2.$

$x^2-2xh+h^2+y^2-2yk+k^2=r^2.$

Rearranging constants together we get.

$x^2-2xh+h^2+y^2-2yk+k^2=r^2.$

Rearranging according to powers of x *and* y and alphabetically

$x^2+y^2-2xh-2yk+h^2+k^2=r^2.$

$x^2 + y^2 - 2xh - 2yk + h^2 + k^2 - r^2 = 0$ (i).

- You notice for a circle the coefficients of x^2 and y^2 are one in the general and expanded form.
- The coefficients of x and y relate to the centre of the circle.

$h^2 + k^2 - r^2$ is a constant.

Hence the equation (i) above can be written as

$x^2 + y^2 - 2xh - 2yk + c = 0.$

This is the expanded general equation of a circle where:

Centre is $(+h, +k)$ and

Radius is $r = \sqrt{(h)^2 + (k)^2 - c}$

And coefficients of x^2 and y^2 are 1

Example

In pairs, find the radius and the centre of the circle given its equation is $x^2 + y^2 - 4x + 2y - 11 = 0.$

Solution

Recall for $x^2 + y^2 - 2xh - 2yk + c = 0$ for centre (h, k) and radius $r = \sqrt{(h)^2 + (k)^2 - c}$

Compare the two equations

$x^2 + y^2 - 4x + 2y - 11 = 0.$

$x^2 + y^2 - 2xh - 2yk + c = 0.$

Hence: $-2h = -4, h = 2.$

$-2k = +2, k = -1.$

The centre is $(2, -1)$

The radius $r = \sqrt{(h)^2 + (k)^2 - c}$

$= \sqrt{2^2 + 1^2 + 11}$.

$= \sqrt{16}$.

$= 4$.

Note: In these methods if the coefficient of both x^2andy^2 are not 1 one is required to make them 1 at the initial step.

The expanded equations of a circle

Just the way a general equation may be expanded in a similar way an equation of a circle may exist as an expanded equation.

For instance the equation of a circle centre (2, 3) and radius 3 units may be written as:-

$(x - 2)^2 + (y - 3)^2 = 3^2$.

$(x - 2)^2 + (y - 3)^2 = 9$.

Expanding the RHS

$(x - 2)(x - 2) + (y - 3)(y - 3) = 9$

$x^2 - 4x + 4 + y^2 - 6y + 9 = 9$

$x^2 - 4x + y^2 - 6y + 4 = 0$

This equation is in the general form of $x^2 + y^2 - 2xh - 2yk + c = 0$

To determine the centre and radius from an expanded equation one may use the general equation method as illustrated in the example above. Alternatively one can complete the squares.

Task

By completing the square show that the centre is (3, -1) and the radius is $\sqrt{2}$ for the circle whose equation is

$x^2 + y^2 - 6x + 2y + 8 = 0$

Exercise 7.4: Solve in groups.

1. Determine the centre and radius of the circles whose equations are.
    a) $x^2 + y^2 = 0.16$
    b) $x^2 + y^2 - 4x + 10y - 20 = 0$
    c) $x^2 + y^2 - 4x - 6y - 15 = 0$
    d) $x^2 + y^2 + 20x + 28y + 144 = 0$
    e) $x^2 + y^2 - 10y - 75 = 0$
    f) $3x^2 + 3y^2 - 12x + 6y = 33$
2. By completing the square determine the centre and radius of a circle whose equation is
    a) $2x^2 + 2y^2 - 2x - 4y - 10 = 10$
    b) $x^2 + y^2 + 2x - 6y = -9$
    c) $x^2 + y^2 - 4x - 6y + 9 = 0$
    d) $2x^2 + 2y^2 - 16y + 6 = 0$
    e) $x^2 + y^2 - 8x = 13$
    f) $x^2 - 4x + y^2 + 25 = -20$
    g) $(x + 3)^2 + y^2 - 10y + 21 = 0$

3. Determine the centre and radius of the following using any suitable method.
    a) $x^2 + y^2 + 24x + 14x - 207 = 0$
    b) $x^2 + y^2 - 4x - 6y + 4 + 9 = 10$
    c) $x^2 + 8x + y^2 + 6y = 0$

4. The equation of a circle of radius 5 units is $x^2 + y^2 - 6x + 4ky + 20 = 0$. Find.
    a) The centre of the circle in terms of k
    b) The radius of the circle

c) The value of k

Determining the equation of a circle given three points on its circumference

Using simultaneous equations

The general equation of a circle can be expressed as $x^2 + y^2 - 2hx - 2ky + c = 0$ since $-2h \, and - 2k$ are different constants we can use $A = -2h \, and \, B = -2k$

The equation simplifies to

$x^2 + y^2 + Ax + By + c = 0$

Substitution of coordinates means generating simultaneous equations that can be solved for A, B and c.

Example

a) Determine the equation of a circle that has points $X(0,2), Y(2,0) and\, Z(0, \ -2)$on its circumference.

b) Find the centre and radius of this circle.

Solution

Using the general form of equation

$x^2 + y^2 + Ax + By + c = 0$

At $Y, x = 2, y = 0$

Substituting in general equation

$$2^2 + 0 + 2A + 0 + c = 0$$

$$2A + C + 4 = 0 \qquad (ii)$$

You notice the above three substitution form three simultaneous equations

$$2A + C + 4 = 0 \qquad (ii)$$

$$2B + C + 4 = 0 \qquad (i).$$

$$-2B + C + 4 = 0 \qquad (iii).$$

To determine the values of A, B and C one can either use algebraic manipulation or the matrix method.

By algebraic manipulation

Equation (i) and (iii) eliminate B

$2B + C + 4 = 0$ and $-2B + C + 4 = 0$ give $2C + 8 = 0$

$2C = -8, C = -4$

Substituting C in Equation (ii)

$2A + C + 4 = 0, 2A - 4 + 4 = 0$

$2A = 0, A = 0$

Substituting C in Equation (iii)

$-2B + C + 4 = 0$

$-2B + \; -4 + 4 = 0$

$-2B = 0, B = 0.$

We obtain $A = 0$, $B = 0$ and $C = -4$.

Substituting the constant on general equation

$$x^2 + y^2 + Ax + By + c = 0$$
$$x^2 + y^2 - 4 = 0$$
$$x^2 + y^2 = 4$$

a) Expressing the equation in general form of $x^2 + y^2 = r^2$ *or* $(x - h)^2 + (y - k)^2 = r^2$ to determine centre and radius we notice that

$x^2 + y^2 = 4$

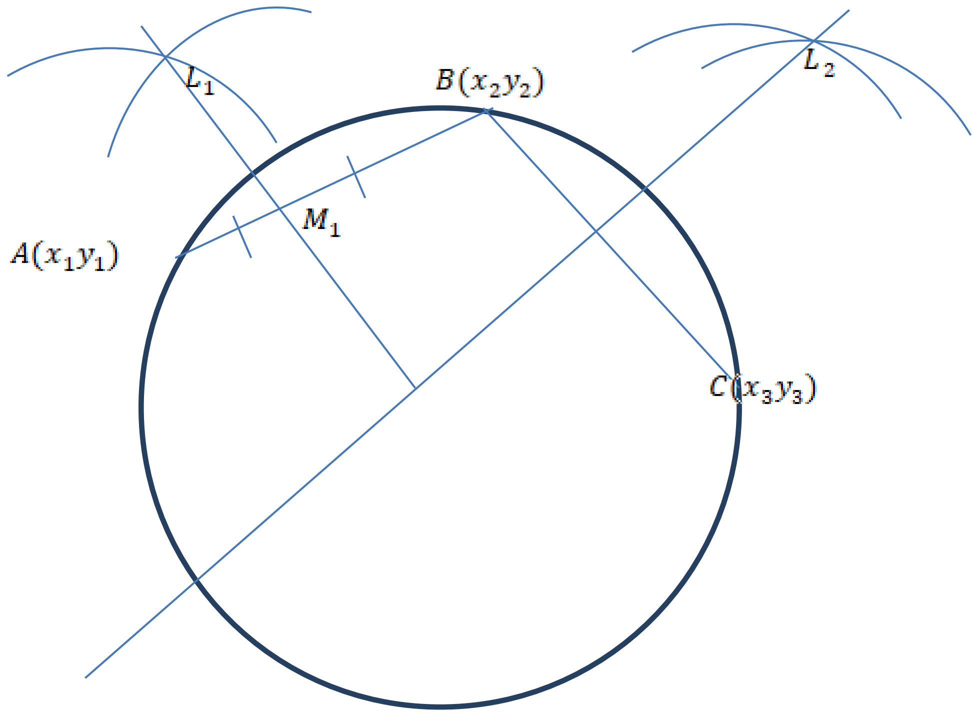
$x^2 + y^2 = r^2$ so centre is (0, 0) and radius is $r = \sqrt{4} = 2\ units.$

Using geometrical method

From geometry we learn that when two points on the circumference are joined they form a chord. The perpendicular bisector of the chord passes through the centre of the circle. This implies that all perpendicular bisectors of chords meet at the centre of the circle. At the center of the circle the perpendicular bisectors also form simultaneous equations.

Example 1

In groups of four study *figure 7.4* and answer the questions that follow. Points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are on the circumference and lines L_1, and L_2 are perpendicular bisectors of lines AB and BC respectively.



a. Write expression for the midpoints in the figure.
b. Write expression for the gradients of the lines AB and BC .
c. What is the relationship between the gradient of the chord and the perpendicular bisector?
d. How useful are the gradient of these lines to the determination of the equation of the circle?

Solution

You notice that if A, B and C have defined coordinates you can calculate the mid-point M_1 and M_3.

$M_1(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

$M_2(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2})$.

Since the coordinates of A, B and C are known we can work out the gradient of lines AB and BC

$$Gradient\ of\ AB = \frac{y_2 - y_1}{x_2 - x_1}$$
$$Gradient\ of\ BC = \frac{y_3 - y_2}{x_3 - x_2}$$

- You notice that line AB and L_1 are perpendicular while line BC and L_2 are perpendicular. Since the product of the gradients of perpendicular lines is – 1 it is possible to determine the gradient of the lines L_1 and L_2
- Using the calculated gradients and midpoints one can find the equation of the lines L_1 and L_2 their intersection point is the centre of the circle
- Using the coordinates of the centre and any one of the points A, B or C one can calculate the equation of the circle.

Exercise 7.5: Work in pairs.

Using any suitable method, determine the equation of a circle that passes through the following points. Sketch the circles. Leave your answer in general form.

1. $(3,4), (5,0)$ and $(0,5)$
2. $(2,0), (4,-5)$ and $(-1,-3)$
3. $(2,-1), (-1,2)$ and $(5,2)$

4. $(1,0), (0,1)$ and $(-1,0)$
5. $(9,4), (8,7)$ and $(4,9)$
6. $(4,-2), (0,0)$ and $(9,3)$
7. $(-2,3), (2,3)$ and $(0,1)$
8. $(-2,-3), (0,-1)$ and $(2,-3)$

Tangent to a circle

A tangent to a circle contacts the circle at only one point on its circumference, just as a tangent does to acurve.

At the contact point of a circle to its tangent there exists a normal that is perpendicular to the tangent.

Figure 7.5 below shows the relationship of a circle and a tangent.

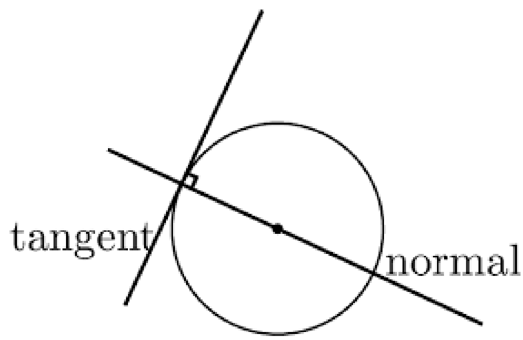


Figure 7.5

You notice the part of the normal to the circle is a radius of the circle. The tangent and the normal radius are perpendicular. The product of their gradient is -1.

$M_{tangent} \times M_{radius} = -1$

$M_t \times M_r = -1$

It is hence possible to calculate

i. The equation of tangent to a circle, given the point of contact to the circle or a point of tangent away from the circle.
ii. The length of a tangent to a circle from a given point.

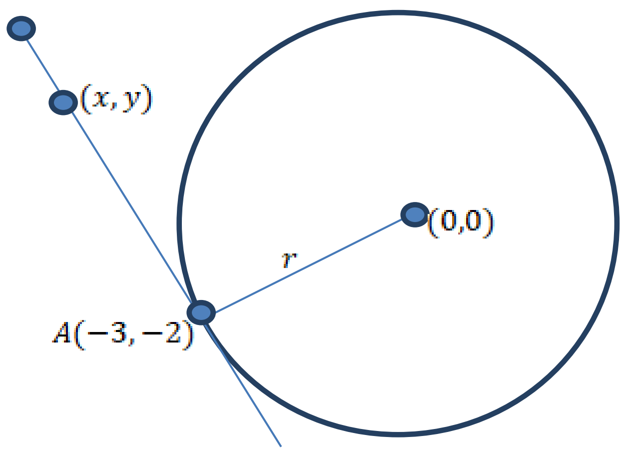
The equation of a tangent of a circle

Given the equation of the circle and a point on its circumference.

Example 1

In pairs, find the equation of a tangent to the circle $x^2 + y^2 = 13$ at a point $A(-3, -2)$

Note the circle has centre (0, 0) and hence the sketch for the diagram is Tangent



$$Gradient\ of\ normal\ radius = \frac{0-(-3)}{0-(-2)} = \frac{3}{2}$$

Gradient of tangentM_t is obtained by

$M_t M_r = -1$ (r and tangent are perpendicular)

$\frac{3}{2} M_t = -1.$

$M_t = -\frac{2}{3}.$

Tangent pass through $(-3, -2), (x, y)$ *and has gradient* $-\frac{2}{3}$

Hence: $\frac{y-2}{x-3} = -\frac{2}{3}$

$$\frac{y+2}{x+3} = -\frac{2}{3}$$

$$y + 2 = -\frac{2}{3}x \pm 2$$

$y = -\frac{2}{3}x - 4$ This is the equation of the tangent

Given a point, the equation of the circle and the length of the tangent

Any external point of a circle always has two tangents. Each of the tangents are normal to the radius at a point of contact as shown in the figure 7.6.

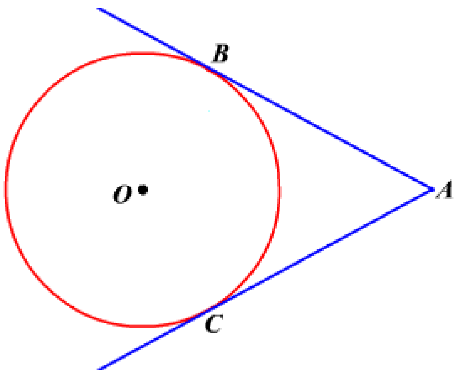


Figure 7.6

If the equation of the circle is defined, its radius and centre can be determined. The distance OA can also be calculated using Pythagoras' theorem.

Since we have right angled triangles OBA and OCA whose hypotenuse and base can be calculated it is possible to calculate the length of AB and AC and their equations using the general point (x, y)

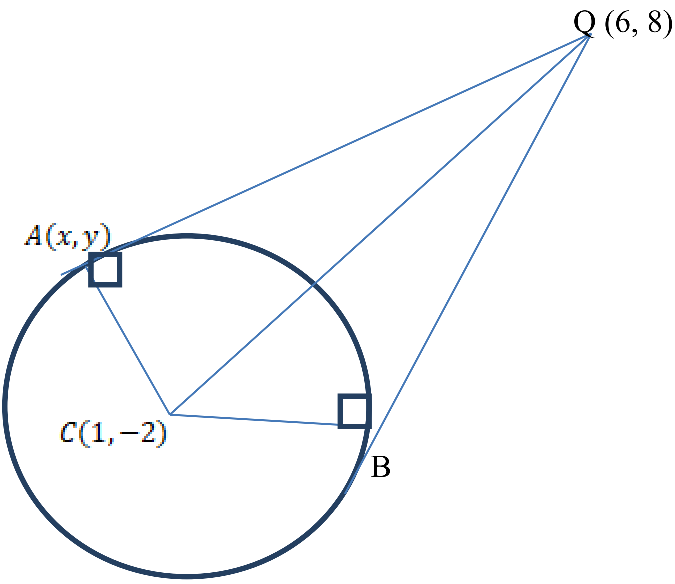
Since triangles OBA and OCA are congruent, the length of the two tangents are equal.

Example 2

A circle has the equation $(x+1)^2+(y-2)^2=16$ and a point Q (6, 8) is away from the circle.

Calculate the length of the tangent from Q to the circle.

Solution



The circle has centre (1,-2) and radius = 4

The distance $CQ=\sqrt{(8-(-2))^2+(6-+1)^2}$

$$CQ=\sqrt{125}.$$

Using Pythagoras' theorem

$(CQ)^2=(AC)^2+AQ^2$

$125=16+AQ^2$

$(AQ)^2=\sqrt{125-16}$

$(AQ)^2 = \sqrt{109}$

The length of both tangents is hence $\sqrt{109}$

Exercise 7.6: Work in pairs.

1. Find the equation of the tangent to the circle $(x-1)^2 + (y+1)^2 = 10$ at the point$(-2,-1)$.
2. Determine the equation of the tangent to the circle $x^2 + y^2 = 13$ from the point (-3,-2).
3. Prove that the line $y = 3x + 10$ is a tangent to the circle $x^2 + y^2 = 10$.
4. Determine the length of a circle $x^2 + y^2 - 6x + 2y + 8 = 0$ from the point (6, -2).
5. Find the length of the tangent drawn from A (4, 5) to the point of contact with circle $(x-3)^2 + (y+5)^2 = 20$.

UNIT 8: KINEMATICS

In kinematics, we study the motion of objects without reference to forces that cause this motion.

We discuss the parameters involved in the motion, such quantities include distance, speed, displacement, velocity and acceleration.

Distance (D)
This is the length between two points or places. It has no reference to the direction of motion.

Displacement (S)
This is the distance from one point to another. It places emphasis on the direction of movement. For instance, the distance from A to B and B to A is the same but the displacement is different since it includes the direction. If A is vertically above B it is very important to specify movement since the other quantities that affect it mainly depend on the direction of movement.

Displacement is a vector quantity while distance is a scalar quantity.

Speed (s)
Speed refers to how fast something moves. It is measured as distance moved over time.

Speed = $\frac{distance}{time}$ $\quad s = \frac{d}{t}$

Speed = $\frac{change\ in\ displacement}{change\ in\ time}$ $\quad s = \frac{dx}{dt}$

Velocity

Velocity is speed with direction. Because the direction is important velocity uses displacement instead of distance.

Velocity = $\frac{displacement}{time}$ $v = \frac{s}{t}$ or $v = \frac{change\ in\ dis\ placement}{change\ in\ time}$ $v = \frac{ds}{dt}$

From $s = \frac{d}{t}$ $d = vt$

Acceleration

Acceleration is rate of change of velocity per unit time

Acceleration= $\frac{velocity}{time}$ $a = \frac{v}{t}$ acceleration = $\frac{change\ in\ velocity}{change\ in\ time}$ $a = \frac{dv}{dt}$

Exercise 8.1: Work in pairs

1. A particle travels for 200 km in 5 hrs. Determine its speed.
2. An aeroplane moves at an avarage speed of 800 km/hr for 2 hrs . calculate the distance covered by the plane.
3. A car passed a point A at a speed of 80 km/hr and at point C it was moving at a speed of 100 km/hr. Calculate the average acceleration if it took 2 hrs to travel from A toB.

Distance-time graphs

These are graphs that describe how a particle moves with respect to time. It explains where an object is relative to a point after some given time. Figure 8.1 below shows the distance tavelled by an object against time in minutes.

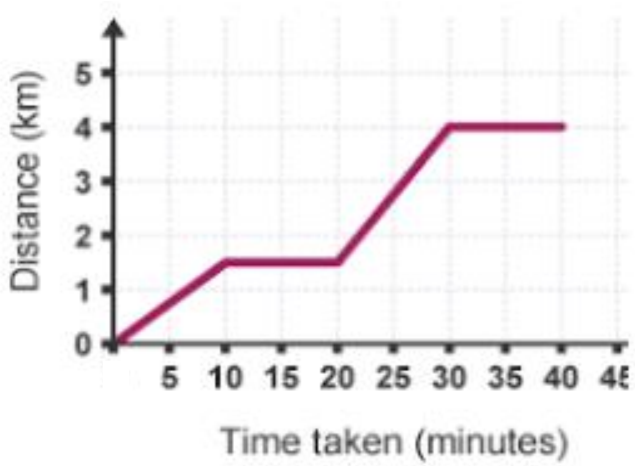


Figure 8.1

The graph shows that the particle started moving at 0 minutes and moved a total of 4 km in 30 minutes.

The particle was at rest between 10 minutes and 20 minutes and from 30 minutes to 40 minutes.

The gradient of the distance time graph represents $\frac{change\ in\ distance}{change\ in\ time}$, which is the speed. If the direction is included then the distance is displacement and the gradient is velocity.

From the graph the velocity of the particle between 20 minutes and 40 minutes is:

Distance at 30 minutes = 4 km

Distance at 20 minutes = 1.5 km

Change in distance = 2.5 km

Change in time = 10 minutes

Hence: speed = $\frac{changeindistance}{changeintime} = \frac{\Delta d}{\Delta t} = \frac{2.5km}{10min}$ = 0.25 km/min.

If the gradient of the distance time graph is positive, it represents an increase in the distance from a given point (i.e. the start). If the gradient is negative it represents a particle moving back to the starting point. Figure 8.2 represents such a motion.

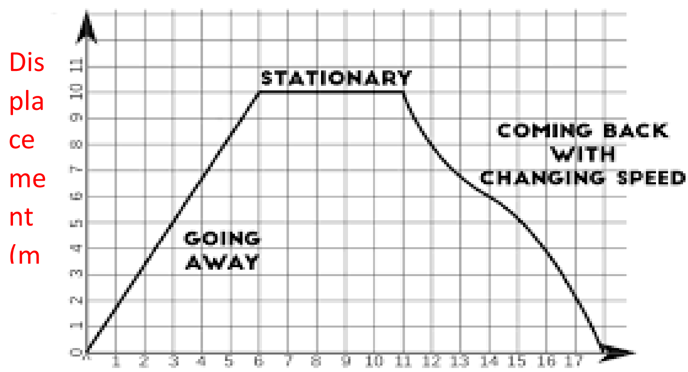


Figure 8.2

From the figure, moving away is at a constant velocity but coming back the velocity is changing since the graph is not a straight line.

Velocity-time graph

This is a graph of velocity against time.

The gradient of the velocity time graph is the acceleration, since it represents the change in velocity against time.

If velocity increases with time the change in motion is called acceleration, $a > 0$

If velocity decreases with time the change in motion is called deceleration, $a < 0$.

A particle moves from one point to another, the velocity at given instances is recorded in table 1.1 below.

Table 1.1

Time	1	2	3	4	5	6	7	8	9	10
velocity	2	4	6	8	8	8	8	8	8	8

The velocity-time graph of this motion is shown in fig 8.3 below.

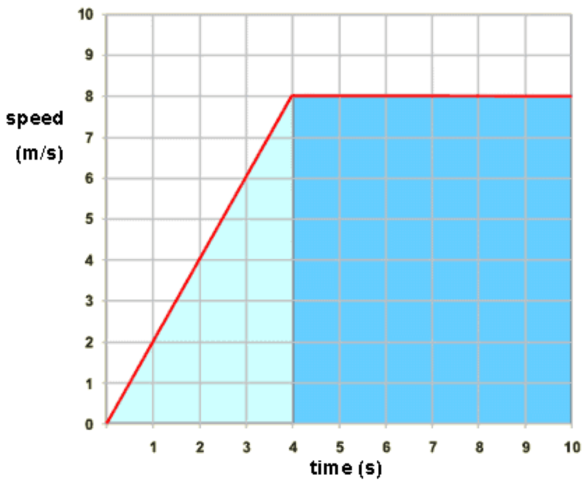


Figure 8.3

The gradient of the graph for the first four seconds is

$a = \frac{dv}{dt} = \frac{(8-0)\, m/s}{(4-0)s} = 2 \text{m/s}^2$

It is the acceleration

The area of the graph for the time from 4 seconds to 10 sec is.

$= t \times v = (10-4) \times 8 = 6 \times 8 = 48\, m$

Notice that the area under a velocity-time graph is velocity multiplied by time and is the distance.

The gradient of the velocity time graph describes the rate of change of velocity: if it a line with positive gradient it is acceleration, if it is a line with negative gradient it is deceleration and if it is a horizontal line it represents movement at constant velocity. Figure 8.4 below illustrates these scenarios.

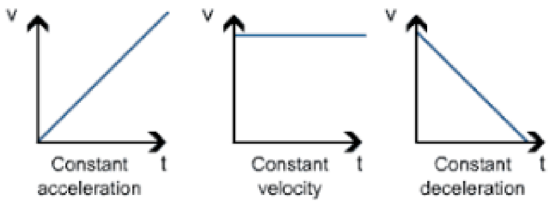


Figure 8.4

Exercise 8.2: Work in groups.

1. Figure 8.5 bshows a graph of velocity against time made by a motorist. Study it and answer the questions that follow.

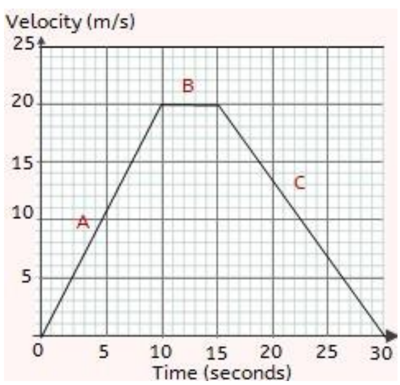


Figure 8.5

(a) Describe the motion over intervals A, B and C
(b) Calculate the acceleration in each interval A, B and C
(c) Calculate the total distance moved by the particle.

2. Calculate the distance covered by an object whose motion is illustrated by the graph in figure 8.6.

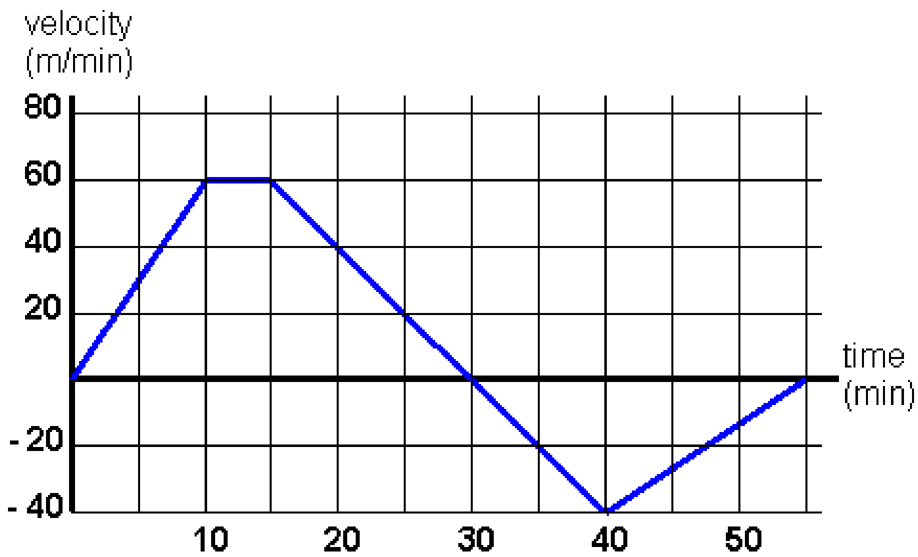
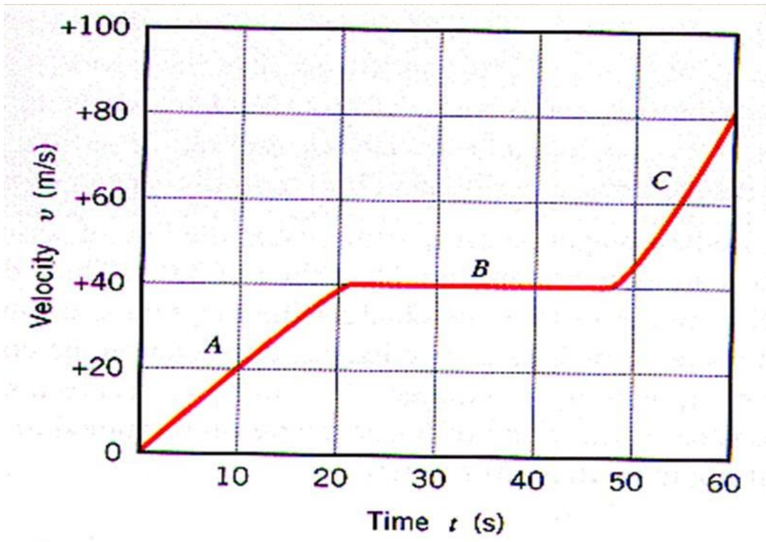


Figure 8.6

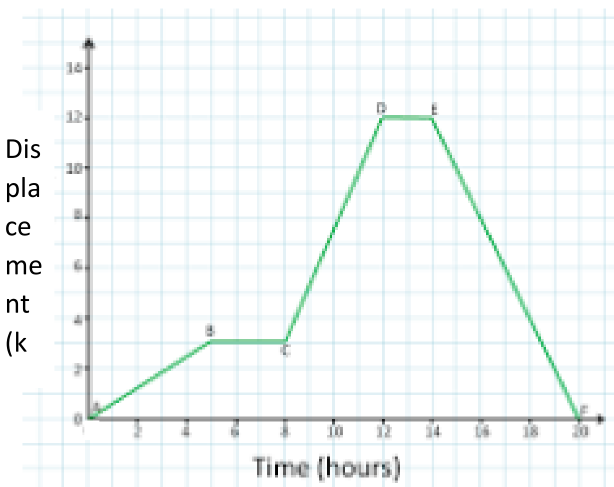
3. Draw the graph showing the movement of a motorist who started from rest, accelerated to a velocity of 80 km/hr in 30 minutes. After 30 minutes he travelled at constant velocity for 30 minutes before decelerating to stop within 5 minutes. Calculate the total distance moved.
Hint: sketch a graph.

4. The motion of a plane is illustrated by the graph in figure 8.7 below.



a. Describe the motion in part A
b. Calculate the acceleration in intervals
   - i) A
   - ii) B
c. Calculate the total distance travelled by the plane in the first 60 seconds.

5. The graph in the figure 8.8 below shows the distance-time graph for a particle moving from a point to the furthest point B.



Calculate:

(a) The total distance moved by the particle
(b) The displacement moved by the particle
(c) The velocity of the particle from 8hrs to 10 hrs.
(d) The velocity of the particle from in the first 2 hours.

Linear motion

Movement in a straight line is referred to as linear motion. This motion can be expressed in terms of distance, velocity and acceleration

There are three main linear motion formulas.

Acceleration is the rate of change in velocity.

$$initial\ velocity = u$$

$$final\ velocity = v$$

$Change\ in\ velocity = v - u$

If this change is done in time t seconds. Then acceleration (a) is expressed as

$$a = \frac{v - u}{t}$$

so $at = v - u$

Making v the subject $v = u + at$

Hence: final velocity= initial velocity + acceleration × time

Example

A particle started from rest and accelerated at a rate of 10 m/s^2 for 6 seconds. Find the final velocity.

Solution

$v = u + at$

$u = 0\ m/s,\ a = 10\ m/s^2,\ t = 6s$

$v = 0 + 10 \times 6 = 60 m/s$

The equation of distance

Consider a particle that starts moving with:

initial velocity= u

final velocity= v

average velocity= $\frac{initial\ velocity + final\ velocity}{2} = \frac{u+v}{2}$

Since distance = velocity × time $s = \left(\frac{u+v}{2}\right)$ t

But $v = u + at$ substituting in s and simplifying.

$$s = \left(\frac{u+u+at}{2}\right)t = \left(\frac{2u+at}{2}\right)t = (u + \frac{1}{2}at)t$$

$$s = ut + \frac{1}{2}at^2$$

The equation of velocity without time is:

$v = u + at$

Squaring both sides $v^2 = (u + at)^2$

Expanding $v^2 = (u + at)\ (u + at)$

$$v^2 = u^2 + 2atu + (at)^2$$

$$v^2 = u^2 + 2atu + a^2t^2$$

$$v^2 = u^2 + 2a(ut + \frac{1}{2}at^2)$$

But $s = ut + \frac{1}{2}at^2$

$$v^2 = u^2 + 2as$$

Hence the square of the final velocity is equal to the square of the initial velocity added to acceleration × distance.

Example 1

A particle accelerates from rest at an acceleration of 20 m/min² for 2 minutes. Calculate the distance covered.

Solution

$$s = ut + \frac{1}{2}at^2$$

$$u = 0 \text{ m/min}, t = 2 \text{ min}, \quad a = 20 \text{ m/min}^2$$

$$s = 0 \times 2 + \frac{1}{2} \times 20 \times 2^2 = 400 \text{ m}$$

Example 2

A jet started at a velocity of 800 km/h and accelerated at a rate of 100 km/h to a final velocity of 1000 km/h. calculate the distance travelled by the jet.

Solution

$v^2 = u^2 + as$

$u = 800 \text{ km/h}, v = 1000 \text{ km/h}, a = 100 \text{ km/h}^2$

Substituting in $v^2 = u^2 + as$

$1000^2 = 800^2 + 100s$

$1\,000\,000 = 640\,000 + 100s$

$$100s = 1\,000\,000 - 640\,000 = 360\,000$$

$$s = \frac{360\,000}{100} = 3600 \text{ m}$$

Exercise 8.3: Work in groups.

1. A car uniformly accelerates from a velocity of 10 m/s to a velocity of 15 m/s in a time of 20 sec. find the distance travelled by the car.

2. A motor cyclist reduces speed from 10m/s to 4 m/s over 28 m. Assuming the deceleration is constant, determine the deceleration and how long the cyclist will travel on his motor cycle before he comes to rest.
3. A car uniformly accelerates from a velocity of 10 m/s to a velocity of 30 m/s in a time of 25 seconds. Calculate the distance travelled.
4. A car reduces speed uniformly from 8 m/s to 5 m/s over 150m. Assuming the deceleration continues, how long does it take to come to rest?
5. For 20 seconds a particle uniformly accelerates from a velocity of 20 m/s to 50 m/s. Find its acceleration.
6. A motorist uniformly reduces his speed from 20 m/s to 5 m/s over 200m. Assuming the deceleration continues calculate:
   (a) the deceleration
   (b) the time taken to come to rest
   (c) the distance taken to come to rest
7. A train starts from rest. After 15 sec its velocity is 20 m/s .how long does it take to travel 200m?
8. A car travelling at a speed of 25 m/s applies brakes to pass over a bump. Its speed reduces uniformly to 10 m/s after 2 minutes. How far would the car travel at the same deceleration to come to a stop?

Movement under gravity

When an object is falling freely, it is subject to an acceleration due to gravity. The value of acceleration varies across the earth. However, the value usually used is $9.8 m/s^2$ towards the earth, it is denoted as g meaning acceleration due to gravity. Considering the three equations of motion

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$v^2 = u^2 + 2as$

On substituting the value of a with $-g$ the **free fall equations** are

$$v = u - gt$$

$$s = ut - \frac{1}{2}gt^2$$

$$v^2 = u^2 - 2gs$$

Example 1

A stone is thrown vertically upwards with an initial velocity of 29.4 m/s.

(a) What is the maximum height?
(b) What is the time taken to reach this height?

Solution

a) $v^2 = u^2 - 2gs$

$u = 29.4\ m/s \quad g = 9.8\ m/s^2 \quad v = 0\ m/s.$

$0 = 29.4^2 - 2(9.8)s$

$$s = \frac{29.4^2}{2(9.8)} = 44.1 \text{ m}$$

b) $v = u - gt$

$0 = 29.4 - 9.8t$

$$t = \frac{29.4}{9.8} = 3 \text{ sec}$$

Example 2

A stone is dropped down from the top of a building. It reaches the bottom of the building in 3.5 seconds. How tall is the building?

Solution

$u = 0$ m/s, $g = 9.8$ m/s^2, $t = 3.5s$

$v = ?$

$$s = ut + \frac{1}{2}gt^2$$

$$s = 0 \times t + \frac{1}{2} 9.8 \times 3.5^2 = 60.0 \text{ m}$$

The building is 60 m tall.

Exercise 8.4: Work in groups.

1. A stone is thrown vertically upwards with a velocity of 5 m/s. What is the maximum height of the stone in the air?

2. A bullet is projected vertically upwards to a maximum height of 154.34 m. Find the initial velocity of the bullet.

3. A ball is dropped down a well. It reaches the bottom of the well in 3.2 seconds. How deep is the well?

4. A watch is dropped from the top of a building 122.5 m high. How long does it take for the watch to reach the bottom of the building?

5. A stone is thrown into the air with a velocity of 8 m/s. How long does the stone take to reach its maximum height?

6. A bullet is fired vertically upward and attains a maximum height of 800 m. Find the initial velocity of the bullet.

7. A bag of cement falls from the top of a wall 45 metres high. Find the bag's speed just before it hits the ground.

8. A stone is thrown vertically upwards with a velocity of 15 m/s from the top of a building 20 metres high. The stone lands on the ground below. Find the time of flight for the stone.

9. A rocket is projected vertically upwards from the ground. It runs out of fuel at a velocity of 52 m/s and a height of 35 m. From this point on it is subject only to acceleration due to gravity. Find the maximum height of the rocket.

10. A mason is vertically up a tower at a certain height and accidently drops his hammer. The man ascends a further 45 m and drops a pair of pliers. The pair of pliers takes 1 second longer than the hammer took to reach the ground. Find:
    a. the height above the ground at which the worker dropped the hammer.
    b. the time it took the pair of pliers to reach the ground.

UNIT 9: VELOCITY

Components of velocity vertically and horizontally

Unlike linear motion where particles move in a straight line most of the time particles move in two or three dimensions. The movement in two dimensions is expressed differently from one dimensional motion.

In this unit we will describe motion in two dimensions. For instance when a ball is hit it moves both horizontally and vertically by travelling in a diagonal path at a velocity (v) as shown in figure 9.1.

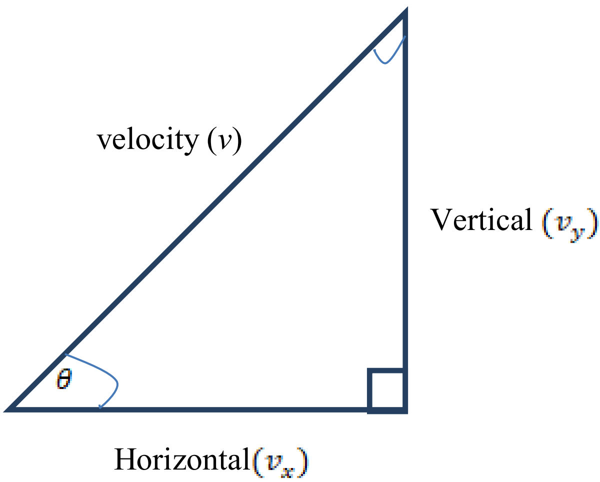


Figure 9.1

These velocities are expressed as vectors to point out the direction of movement. For instance if (v_y) is positive the object moves up, if negative it moves down and if (v_x) is positive object moves to the right and if negative it moves to the left. Since the direction is important the velocity can be written as a vector $v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$.

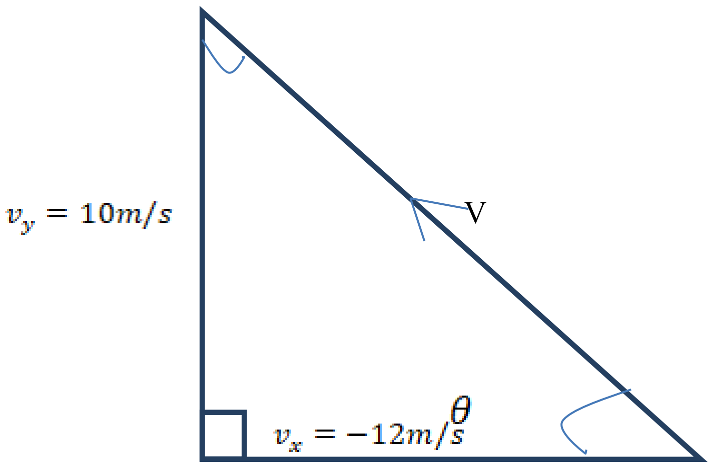
The magnitude of the velocity can be calculated using Pythagoras' theorem $v^2 = {v_x}^2 + {v_y}^2$, $v = \sqrt{{v_x}^2 + {v_y}^2}$.

Example 1

A ball is hit such that the components of velocity are
$v_x = -12m/s$ and $v_y = 10m/s$

a) Illustrate these movements on a sketch.
b) Calculate the angle of projection of the ball.
c) What is the magnitude of the initial velocity (v)?

Solution



b) $\tan\theta = \frac{10}{-12}$

$\theta = \tan^{-1}\left(\frac{10}{-12}\right)$

$\theta = 39.8^0$ $to\ 1d.p..$

c)$v^2 = {v_x}^2 + {v_y}^2 = 100 + 144$

$v = \sqrt{244} = 15.6$ m/s $to\ 1d.p.$

Example 2

A child kicks a soccer ball off of the top of a hill. The initial velocity of the ball is 15.0 m/s horizontally. After 5.00 s, what is the magnitude of the velocity of the ball?

Solution

The velocity of the ball after 5.00 s has two components. Once these two components are found, they must be combined using vector addition to find the final velocity. The ball was kicked horizontally, so v_{xo} = 15.0 m/s, and v_{yo} = 0.0 m/s. The x component of the velocity after 5.00 s is:

$v_x = v_{xo}$

and the y component is:

$v_y = v_{yo} - gt$

$v_y = (0.0\ m/s)-(9.80\ m/s^2)(5.00\ s)$

$v_y = -49.0\ m/s$

In projectile motion problems, up is defined as the positive direction, so the *y* component has a magnitude of 49.0 m/s, in the down direction.

To find the magnitude of the velocity, the x and y components must be added with vector addition:

$v^2 = v_x^2 + v_y^2$

$v^2 = (15.0\ \text{m/s})^2 + (-49.0\ \text{m/s})^2$

$v^2 = 2626\ (\text{m/s})^2$

$\therefore v = 51.24\ m/s$

The magnitude of the velocity is 51.24 m/s.

Though it was not asked for in the question, it is also possible to find the direction of the velocity as an angle. If the horizontal direction is 0.0 radians, the angle can be found with the equation:

$$\tan\theta = \frac{opposite}{adjacent}$$

$$\tan\theta = \frac{v_y}{v_x}$$

$$\tan\theta = \frac{-49.0\,m/s}{15.0\,m/s}$$

$tan\theta$ = -3.267

$$\therefore \theta = \tan^{-1}(-3.267)$$

θ = -1.274 radians

θ = -72.98°

Therefore, the velocity (magnitude and direction) of the ball after 5.00 s was 51.24 m/s, -72.98° down from the horizontal.

Example

A toy rocket is launched in a flat field, aimed at an angle 60.0° up from the horizontal (*x*) axis. Its initial velocity has a magnitude of 20.0 m/s. How much time does the toy rocket spend the air, and how far from its launch point does it land in the field?

Solution

The first thing that must be found to solve this problem is the initial velocity in the x and y directions. The velocity that is given has both x and y components, because it is in a direction 60.0° up from the horizontal (x) direction. The initial velocity can be broken down using an equation relating the sine and cosine:

$1 = cos^2\,\theta + sin^2\,\theta$

We multiply both sides by the initial velocity squared v_o^2:

$(v_o^2)(1) = (v_o^2)(cos^2\ \theta) + (v_o^2)(sin^2\ \theta)$

$v_o^2 = (v_o\ cos\theta)^2 + (v_o\ sin\theta)^2$

The equation for vector addition of the initial velocity components is:

$v_o^2 = v_{xo}^2 + v_{yo}^2$

So, the components of the velocity can be set equal to the parts of the equation above:

$v_{xo}^2 = (v_o\ cos\theta)^2$

$\therefore v_{xo} = v_o\ cos\theta$

and $v_{yo}^2 = (v_o\ sin\theta)^2$

$\therefore v_{yo} = v_o\ sin\theta$

So, the initial velocity in the x direction v_{xo} is:

$v_{xo} = v_o\ cos\theta$

$v_{xo} = (20.0\ \text{m/s})(\cos 60°)$

$$v_{x0} = (20.0\ m/s)\left(\frac{1}{2}\right)$$

$v_{xo} = 10.0\ \text{m/s}$

and the initial velocity in the y direction is:

$v_{yo} = v_o\ sin\theta$

$v_{yo} = (20.0\ \text{m/s})(\sin 60°)$

$$v_{y0} = (20.0\ m/s)\left(\frac{\sqrt{3}}{2}\right)$$

$v_{yo} = 17.32\ \text{m/s}$

To find how long the toy rocket was in the air, use the equation for the vertical distance for the rocket:

$$y = v_{y0}t - \frac{1}{2}gt^2$$

The field is flat, so the rocket will hit the ground again at y = 0.00 m. So, set the left side of the equation equal to that height:

$$0.00\,m = v_{y0}t - \frac{1}{2}gt^2$$

$$0.00\,m = (17.32\,m/s)t - \frac{1}{2}(9.80\,m/s^2)t^2$$

$(-4.90\ m/s^2)t^2 + (17.32\ m/s)t + 0.00\ m = 0$

This has the form of the quadratic equation, with *t* as the variable:

If $at^2 + bt + c = 0$, $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\therefore t = \frac{-(17.32\,m/s) \pm \sqrt{(17.32\,m/s)^2 - 4\,(-4.90\,m/s^2)\,(0.00m)}}{2\,(-4.90\,m/s^2)}$$

$$t = \frac{-(17.32\,m/s) \pm \sqrt{(17.32\,m/s)^2}}{(-9.80\,m/s^2)}$$

$$t = \frac{-(17.32\,m/s) \pm (17.32\,m/s)}{(-9.80\,m/s^2)}$$

There are two solutions for t, depending whether the positive or negative in the ± symbol is chosen. Solving for each:

$$t_+ = \frac{-(17.32\,m/s) + (17.32\,m/s)}{(-9.80\,m/s^2)}$$

$$t_+ = \frac{-0.00\,m/s}{(-9.80\,m/s^2)}$$

$t_+ = 0.00$ s

and

$$t_- = \frac{-(17.32\,m/s) - (17.32\,m/s)}{(-9.80\,m/s^2)}$$

$$t_{-} = \frac{-(34.64\, m/s)}{(-9.80\, m/s^2)}$$

t_{-} = 3.53 s

The first solution is the starting time, and the second is the time at which the toy rocket reaches the ground again. Therefore, the amount of time that the toy rocket spent in the air was 3.53 seconds.

To find the distance from the starting position to the landing position, use the equation for horizontal distance:

$x = v_{xo}\, t$

The time that the toy rocket traveled through the air was just found to be 3.53 *s*. Using that number, the horizontal distance is:

$x = v_{xo}\, t$

x = (10.0 m/s)(3.53 s)

x = 35.3 m

The toy rocket lands 35.3 m away from its launch position.

Exercise 9.1: Work in pairs.

1. A ball is kicked at an angle of 30^0 with a speed of 24.3 m/s.
   a) Illustrate the motion of a sketch.
   b) Calculate the vertical and horizontal components of the velocity.
2. A seagull is flying with velocity $v = \begin{pmatrix} 14.6 \\ -8.62 \end{pmatrix}$ m/s .
   a) What is the magnitude of the velocity of the seagull?
   b) What is angle of the velocity?

3. A ball was thrown at 30° angle and the speed is 20 m/s.

a) What's the minimum speed of the ball in the flight path? Where will that happen?
b) How long will the ball travel before hitting the ground?
c) What is the maximum high that the ball can reach?
d) How far will the ball reach horizontally?
e) What is the final velocity right before hitting the ground?
f) If the angle changes to 60°, how far will the ball reach?

4. A pilot flying a constant 215 km/h horizontally in a low-lying helicopter, wants to drop documents into his contact's open car which is traveling 155 km/h in the same direction on a level highway 78.0 m below. At what angle (to the horizontal) should the car be in his sights when the packet is released?

5. An object is thrown horizontally off a cliff with an initial velocity of 5.0 meters per second. The object strikes the ground 3.0 seconds later.
   a) What is the vertical speed of the object as it reaches the ground?
   b) How far from the base of the cliff will the object strike the ground?
   c) What is the horizontal speed of the object 1.0 second after it is released? [Neglect friction.]

6. Stones are thrown horizontally with the same velocity. One stone lands twice as far as the other stone. What is the ratio of the height of the taller building to the height of the shorter?

7. A tiger leaps horizontally from a 6.5 m high rock with a speed of 4.0 m/s. How far from the base of the rock will she land?

8. A projectile was fired horizontally from a cliff 19.62 m above the ground. If the horizontal range of the projectile is 20 m, calculate the initial velocity v0 of the projectile.

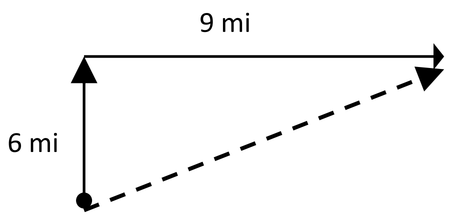
9. A ball projected horizontally with an initial velocity of 20 m/s east, off a cliff 100. meters high. [Neglect air resistance.]
   a) During the flight of the ball, what is the direction of its acceleration?
   b) How many seconds does the ball take to reach the ground?

10. Reflection Question: Mathematically prove that the trajectory of a projectile is actually a parabola (i.e., a quadratic equation).

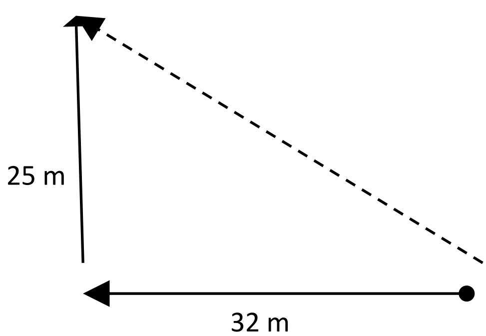
Exercise 9.2: Work in pairs

Find the magnitude and angle of the resultant.

1.



2.

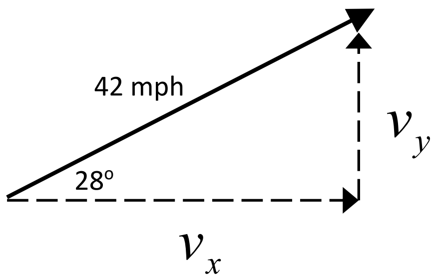


3. Bill drives 14 miles North, then 5 miles East, then 4 miles South.
    a. What is his total distance traveled?

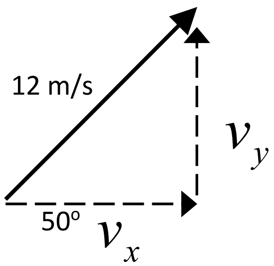
    b. What is Bill's displacement (magnitude and direction).

Find the horizontal and vertical components of each vector.

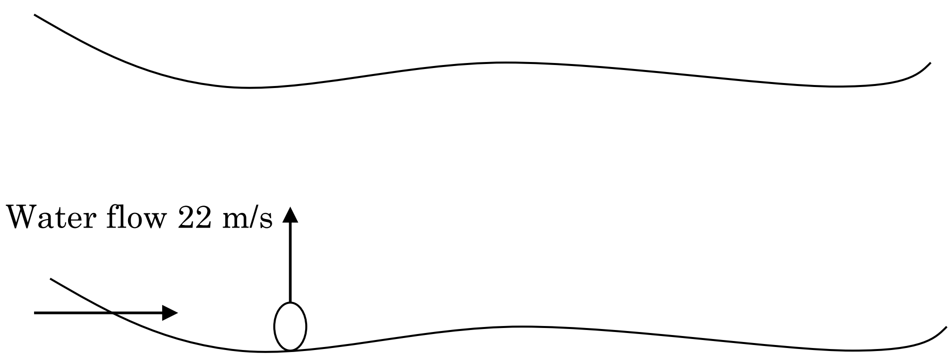
4.



5.



6. A plane taking off is traveling 225 m/s at an angle of 22 degrees with the ground. Calculate the horizontal and vertical components of its velocity.



A small motorboat normally travels at 5 m/s in calm water. It is trying to get across a 56-meter wide raging river that is flowing at 22 m/s.

7. Draw a vector diagram that shows the velocity of the boat.

8. Find the resultant vector for velocity (magnitude and direction)

9. How much time does it take for the boat to reach the other bank?

10. How far downstream is the boat (relative to where it started) when it reaches the opposite bank?

Ted drives 15 miles East then turns 30 degrees to the North and drives 23miles.

11. What distance did Ted travel?

12. What is Ted's displacement (magnitude and direction)?

Hint: Start by drawing a vector diagram then resolve the angled path into components.

UNIT 10: FORCE AND MOMENTUM

Force makes changes to a body's motion. It can change the velocity of an object. The standard international units of force are Newton (N). Newton's First law is **Force = mass × acceleration**

Types of forces

There are many types of forces.

Weight (w)

Weight is the force due to gravity. Weight is experienced as a downwards pull on an object. It is proportional to the mass of the object.

Weight is the product of mass (in kg) and the acceleration due to gravity (*g*) 9.8 m/s².

Example

Calculate the weight of a stone 5kg.

Solution

$$weight = mass \times acceleration\ due\ to\ gravity$$
$$= 5 \times 9.8$$
$$= 49N$$

In a diagram, weight is expressed as downwards force.

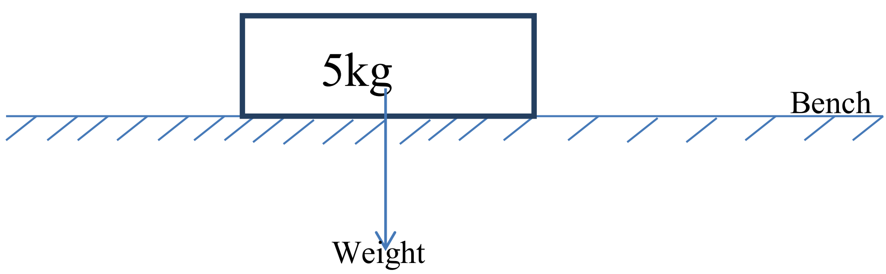


Figure 10.1

Figure 10.1 above illustrates the weight of a stone.

Reaction force

The stone does not fall through the table because the table exerts an equal and opposite force to the weight known as the normal reaction. The normal reaction force acts perpendicular to the surface.

The reaction force to the force of gravity is illustrated in figure 10.2 below.

The reaction force to weight is called the Normal Force (F_N)

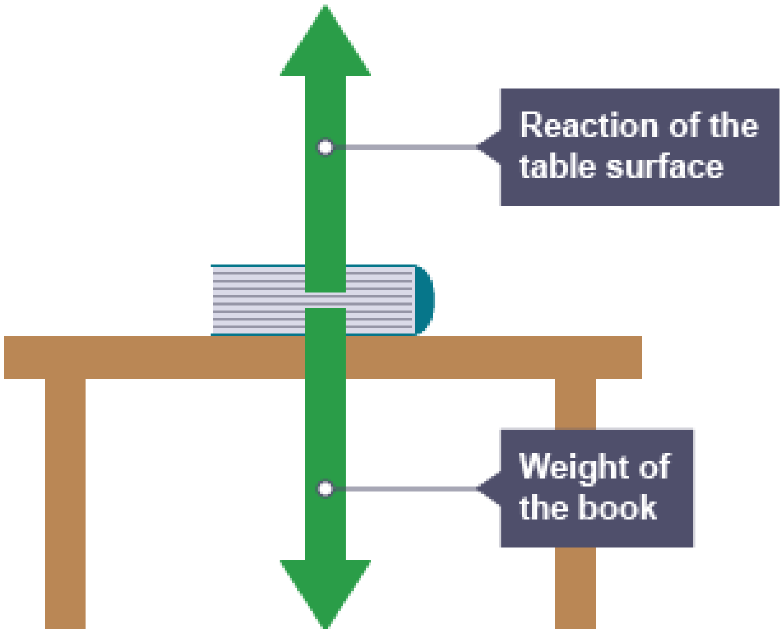
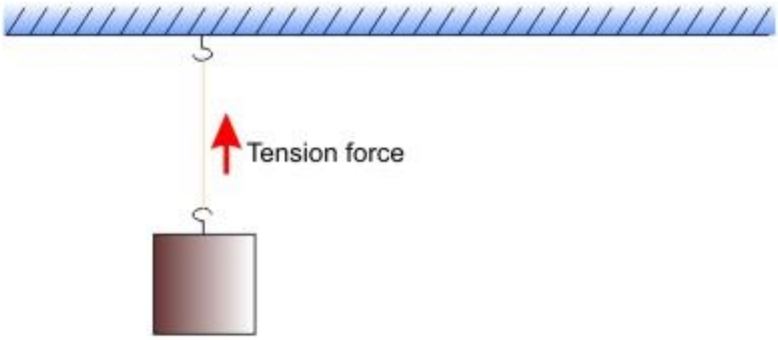


Figure 10.2

Tension (T)

Tension is a force that is transmitted through a string, rope, cable or wire when it is pulled tight by a force acting in the opposite direction. Its symbol is T. It is directed along the direction of the wire and pulls on the object in the opposite direction.

The figure 10.3 below shows tension force.



Tension force T depends on the weight and any net motion

If the stone is moving upwards (Pull) the tension increases, if it is moving downwards the tension decreases

$Tension = weight \pm mass \times acceleration$

$T = mg \pm ma$

Example

A stone 10kg is hanging. Calculate:

i. Tension force if acceleration of the stone is 0 m/s^2
ii. Tension force if the acceleration of stone is $+5m/s^2$ (upwards)
iii. Tension force if acceleration of stone was -5m/s (downwards)

Solution

i. $T = mg + ma$
$= 10 \times 9.8 + 5 \times 0 = 98$ N

ii. $T = mg + ma$
$= 9.8 \times 10 + 5 \times 10 = 148$ N

iii. $T = mg + ma$
$= 10 \times 9.8 - 10 \times 5 = 98 - 50 = 48$ N

Friction force (F_f)

Friction force is the retarding force to a moving object. It comes into play when two bodies are in contact with each other.

Friction force (F_f) depends on the weight of the object and the coefficient of friction(μ). The coefficient of friction depends on the texture of the surfaces of contact between the two bodies.
Figure 10.4 below shows relationship between friction force, weight and applied force on an object.

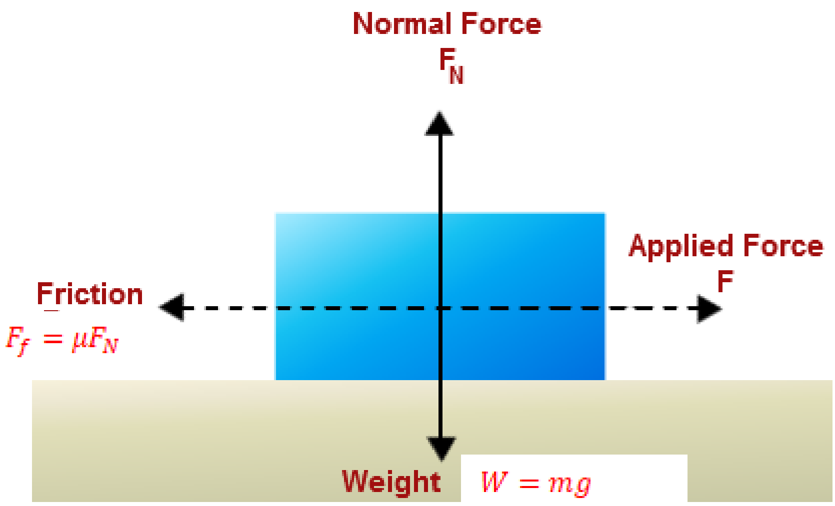


Figure 10.4

F_f – Is friction force

μ – is coefficient of friction

F_N - Is the Normal force

Example

A truck pulls a car 1 tonne. What is the normal force on the car?

a) What is the frictional force between the car and the road if the coefficient of friction is 0.2.

Solution

a) $F_N = weight = mg$

$= 1000 \times 9.8 = 9800\ N$

b) $F_f = \mu F_N = 9800 \times 0.2 = 1960\ N$

Thrust force

Thrust is a force that accelerates a mass in a particular direction. The accelerated mass will cause a force equal in magnitude (assuming the object is moving at constant velocity) but in an opposite direction (drag).

Some of the best illustration for a thrust force is a flying aeroplane, a motorboat moving on water or an object floating on river water, a moving fish and figure 10.5a below illustrate the thrust forces of a moving fish.

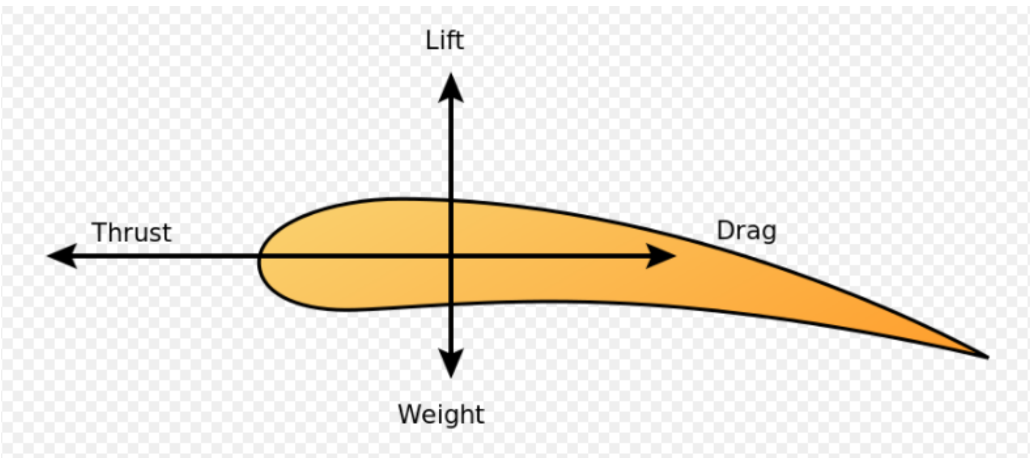


Figure 10.5a

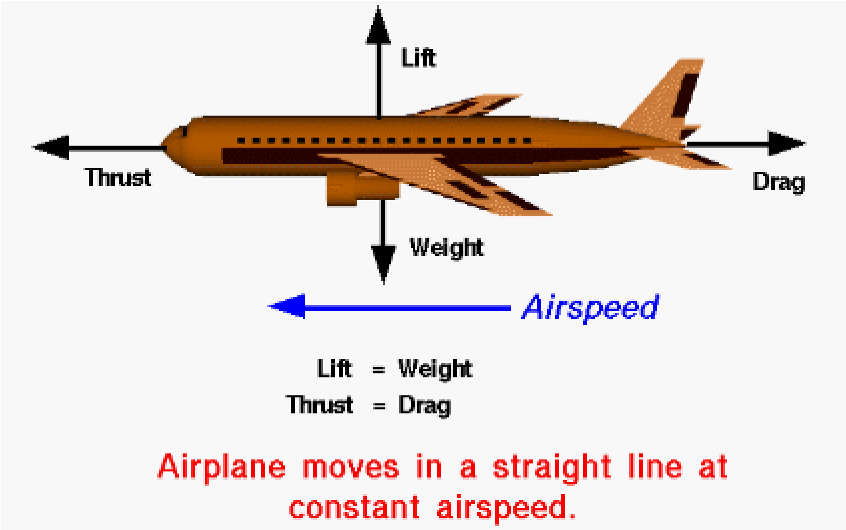


Figure 10.5b

Figure 10.5b shows the thrust force experienced by a flying plane.

From the Newton's law thrust is a product of the mass of moving body (m) × acceleration of the moving body. The net thrust is engine thrust less the drag.

$$Excess(Net)Thrust = Thrust(Engine) - Drag$$

$$F_{(excess)} = ma$$

Example

Calculate the excess thrust of a plane 15 tonnes accelerating at a rate of 200 m/s²

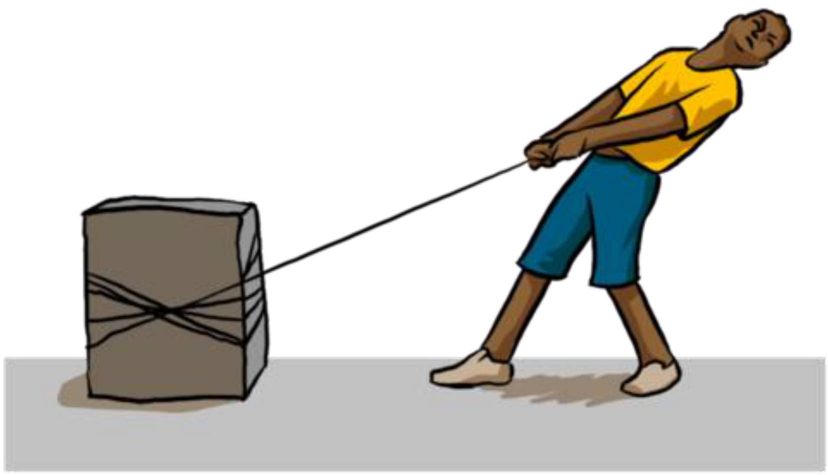
Solution

$$T_{(excess)} = ma$$

$$= 15\,000 \times 200 = 3\,000\,000\,N = 3000\,kN$$

Exercise 10.1: Work in pairs. Assume objects are on a horizontal plane.

1. Calculate the weight of objects with mass
   a) 2 kg
   b) 10 kg
   c) 3 tonnes
   d) 0.25 kg
   e) 300 g
   f) 200 kg
2. Calculate the reaction force on the following objects with mass.
   a) 3 kg
   b) 2.5 kg
   c) 5 tonnes
   d) 2.1 kg
   e) 2.5 kg
3. Calculate the mass of the objects with the following weight
   a) 2 N
   b) 12 N
   c) 1042 N
   d) 0.84 N
   e) 0.98 N
4. A flying machine lifts a package of 3 tonnes upwards at an acceleration of $10m/s^2$. Calculate the tension force.
   Hint: draw a diagram
5. Garang is pulling a solid of 10kg.



   Calculate:
   a) The weight of the box.
   b) The normal reaction force on the box.
   c) The frictional force if the coefficient of friction is 0.3.

Composition of forces

Composition of forces is also called the resultant force. It is the net force of all the forces acting on an object. It is the vector product when two or more forces act on a single object. It is calculated by vector addition of the forces acting on an object.

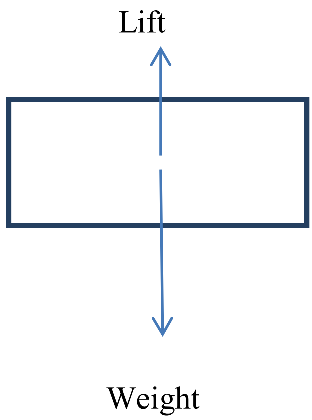
The composition of forces may include force acting on one line, a point or at different lines.

Forces acting in one line can be added

Example

A mason lifts a stone of 20kg with a force of 350N. What is the resultant force with which he will lift the stone? What is he acceleration?

Solution



$weight = 20 \times 9.8 = 196\ N$

$Lifting\ force = 350N$

Assume upward force is positive

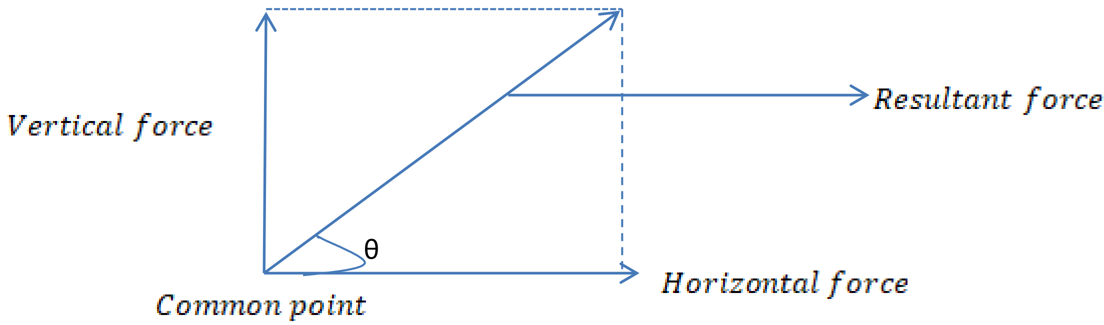
$Resultant\ force = 350N - 196N = 154N.$

He was able to lift it since the resultant force upwards is 154N.

The acceleration is $154 \div 20 = 7.7$ m/s^2

Concurrent forces

These are forces that do not act in one line but all pass through same point



The resultant force vector is the diagonal of the rectangle created by both horizontal and vertical forces and their parallel lines.

The magnitude of resultant force can be calculated by use of Pythagoras' theorem and its direction by trigonometry.

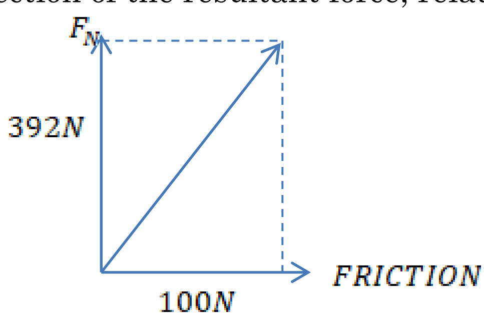
$$F = \begin{pmatrix} F\cos\theta \\ F\sin\theta \end{pmatrix}$$

Example

An object of mass 40kg is moving along the ground, experiencing a frictional force of 100N. Calculate

a) The magnitude of the resultant force of the normal reaction and friction
b) The direction of the resultant force, relative to the ground

Solution



$F_N = weight = 40 \times 9.8 = 392$ N

a) $Resultant\ force = \sqrt{392^2 + 100^2}$

$= \sqrt{163\ 664} = 404.6\ N$ to 1d.p.

b) $\sin\theta = \frac{392}{404.55} = 0.968968$

$\theta = \sin^{-1} 0.968968$

$\theta = 75.7^0$ to 1d.p.

Exercise 10.2: Work in pairs.

1. A crane is lifting a1 tonne bag using a force of 9900N. Calculate the resultant force.
2. A coach is assisting his colleagues on a bench and press with a barbell of 100 kg. The coach provides a force of 70N and his colleagues a force of 920N.
   i) What is the resultant force?
   ii) Did they manage to pull the barbell?
3. The weight of an object is 2200N. A frictional force of 500N is experienced backwards. Find the resultant force of the normal reaction and the frictional force.
   a) What direction is the resultant force?
   b) What is the magnitude of the resultant force?

Resolution of forces

Resolution of forces is separation of a single force into two or more different forces with different directions that when taken together are equivalent to the single force. It is the opposite of composition forces.

Example

An object experiences a resultant force of 120N at an angle of 60° to the x direction. Calculate the components of the force, F_x and F_y

Solution

$F_x = 120\cos 60° = 60$ N

$F_y = 120\sin 60° = 103.9$ N to 1d.p.

Co-planar forces

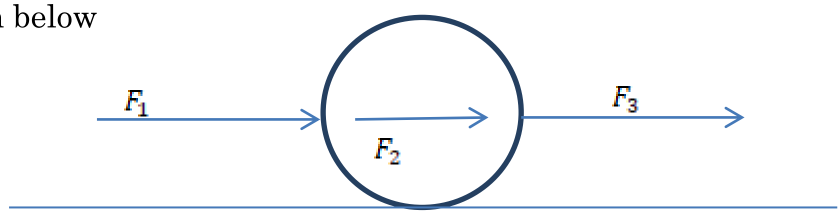
A plane is a flat surface. Co-planar forces act in one plane. These forces may be concurrent, parallel, non-concurrent or non-parallel.

The resultant co-planar forces are determined by the magnitude of the different forces and their direction of action.

Collinear co-planar forces

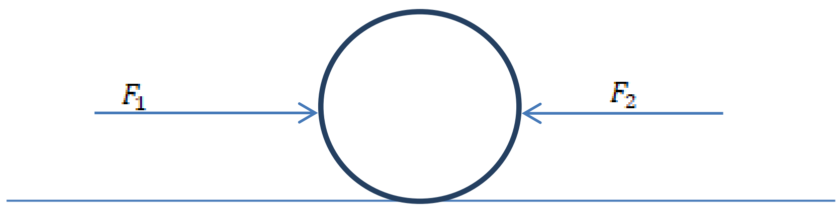
These are forces acting in one line in a given direction.

For instance on pulling a stone and pushing it to same direction as shown below



$F_1 = Push\ force.$
$F_2 = engine\ push.$
$F_3 = pulling\ force.$
The resultant force $F = F_1 + F_2 + F_3$

If a body is pushed from both sides then the resultant force is the difference.



$$Resultant\ force = F = F_2 - F_1$$

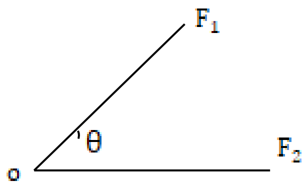
Triangle and polygon of forces

Resultant concurrent co-planar forces

These are forces that act in the same plane, in different directions but intersect at a common point.

Graphical method is easier to calculate resultant forces

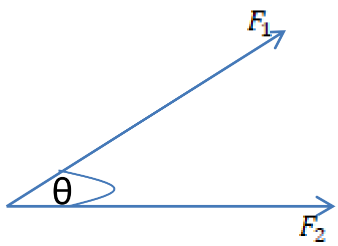
Consider the forces



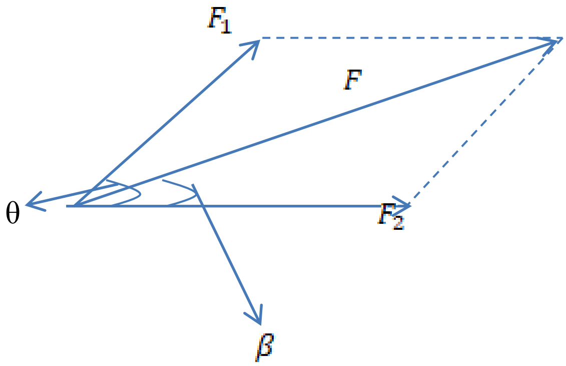
The steps involved are:-

i) Choose a convenient scale for F_1 and F_2
ii) From point O draw vector OF_2
iii) From point O draw vector OF_1 at an angle θ

Consider two forces acting at a point as shown in the figure 10.6 below.



The angle between F_1 and F_2 is θ^0. The resultant force F forms a parallelogram as shown below.

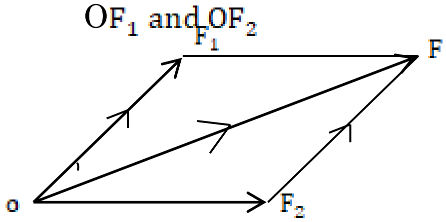


Using trigonometry

$$F = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

$$\beta = \sin^{-1}\left(\frac{F_1 \text{Sin } \theta}{F}\right)$$

iv) Complete a parallelogram $OF_1F_rF_2$ by drawing parallel lines to OF_1 and OF_2

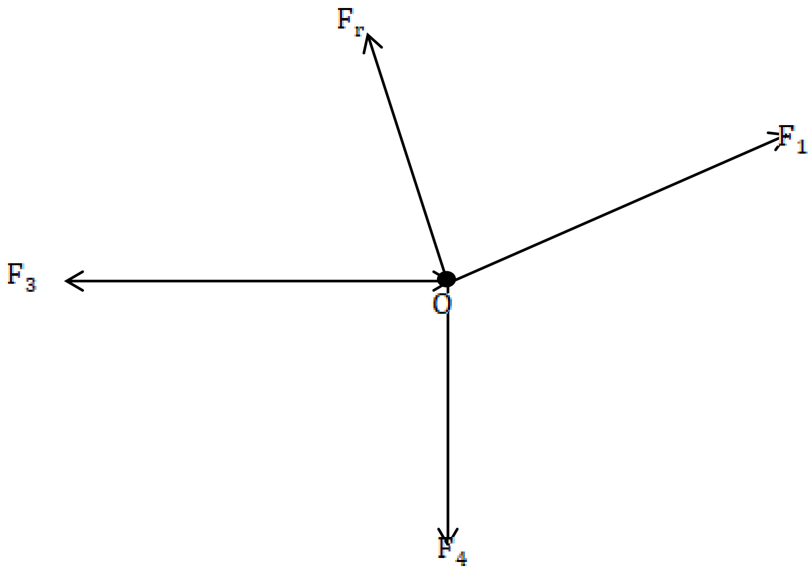


Measure line OF for *F*(resultant force) multiply OF by the scale to find the resultant force.

Resultant forces of more than two forces

When there are more forces we develop a polygon of forces and a scale drawing can be used.

Consider the four forces $F_1 F_2 F_3$ and F_4 in the figure 10.7 below



i) Choose an appropriate scale for F_1, F_2, F_3 and F_4

ii) Taking any point A, draw a vector OF_1 parallel to OF_1 such that

$OF_1 = AB = F_1$

iii) From B, draw a vetor parallel to OF_2 such that BC = $OF_2 = F_2$

iv) From point C, draw CD parallel to OF_3 and $CD = OF_3 = F_3$

v) From point D, draw CE parallel to OF_4 and $DE = OF_4 = F_4$

vi) Join A to E to form a polygon. AE = F which is the resultant magnitude and direction of these forces.

Magnitude of resultant force F = Length AE $\times$ Scale

Exercise 10.3: Work in groups.

Hint: always draw a digram

1. A force of 100N acts at a point at 30^0 to the horizontal. Determine the:
   a) The force in the vertical direction.

b) The force in the horizontal direction.

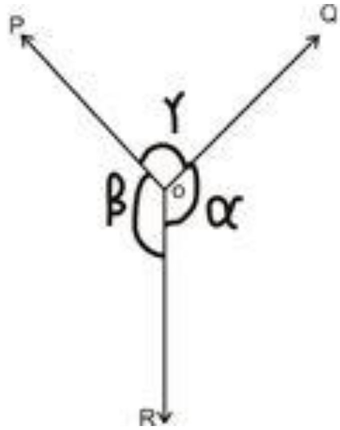
2. A metal block weighing 100N is placed on an inclined plane that makes an angle of 30^0 with the horizontal. Calculate the component of the weight;
   i) Parallel to the inclined plan.
   ii) Perpendicular to the inclined plane.
3. Four co-planar equal forces act from a point O where their bearings are 090^0 for F_1, 336^0 for F_2, 267^0 for F_3 and 189^0 for F_4.
   i) Draw a sketch illustrating these forces.
   ii) Determine the relative magnitude and direction of the resultant force.
4. Three collinear forces of magnitude 200N, 100N and 300N are acting on a rigid body. Determine the resultant forces.
5. Four collinear forces of magnitude 200N to right, 400N to right, 20N to the left and 180N to the left act on a body. Determine the resultant force.
6. A boy lifts a load of 10kg with a force of 100N. He is assisted by a machine with pulling force 20N. Calculate the resultant force.
7. Three forces A, B and C of magnitude 30 kN, 25 kN and 45 kN respectively act at a point. The angle between the forces A and B is 35^o and the force C makes an angle of 130°with the force A in the anticlockwise direction. Find the resultant force F and its direction.
8. Find the resultant of two forces 80 000N and 50 000N act at a point with an angle between them of 60°.

Equilibrium of forces and Lami's Theorem

Resultant force, in its most basic form, is force carrying magnitude and direction resulting from the actions of a given set of forces on a particular point or particle. The resultant force always produces the same effect as the net force generated by all the given forces.

If the resultant force is zero, body is in equilibrium, i.e. there is no net acceleration, so the body will either continue at rest or moving with constant velocity.

Analytical Method for the Equilibrium of Three Forces



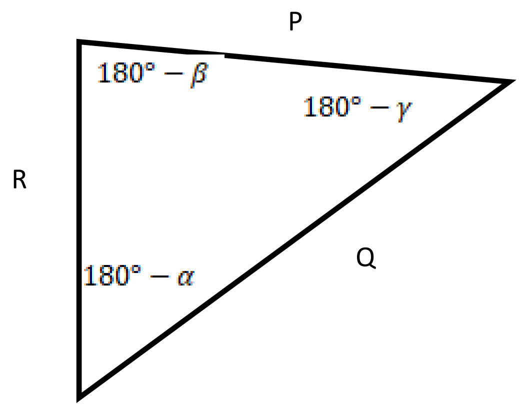
If a set of given forces acting on a body does not produce any alteration in the motion of the body, the forces are in equilibrium.

Lami's Theorem states: "If three coplanar forces acting on a point produce the effects of equilibrium, then each of them are proportional to the sine of the angle between the other two."

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Where P, Q and R are the given forces and α, β and γ are their respective angles as given in the diagram.

Now let's derive the theorem.



The three forces are in equilibrium so form the closed triangle as shown.

Using the sine rule

$$\frac{P}{\sin(180° - \alpha)} = \frac{Q}{\sin(180° - \beta)} = \frac{R}{\sin(180° - \gamma)}$$

From the graph of $\sin\theta$, $\sin(180° - \theta) = \sin\theta$

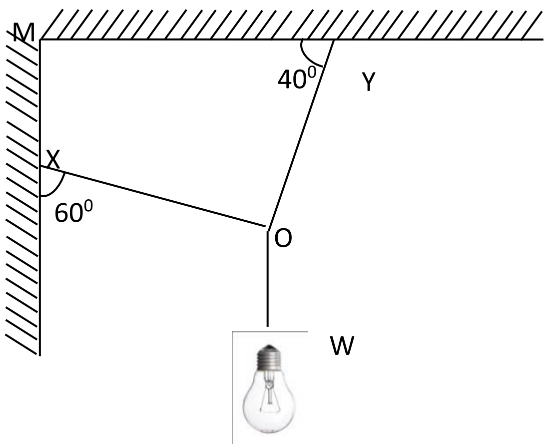
So the expression becomes:

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Which is Lami's theorem.

Example

A light bulb weighing 42N is hanging as shown in the figure 10.9 below.



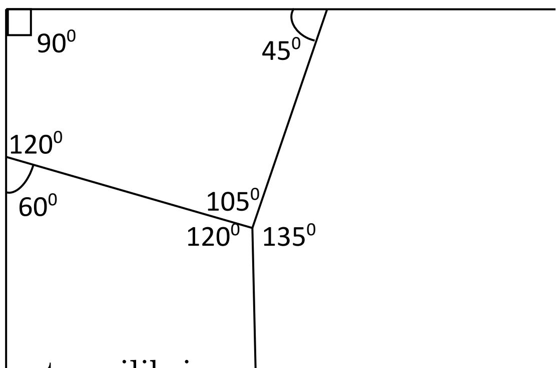
Calculate i) Tension in OX, T_x

ii) Tension in OY, T_y

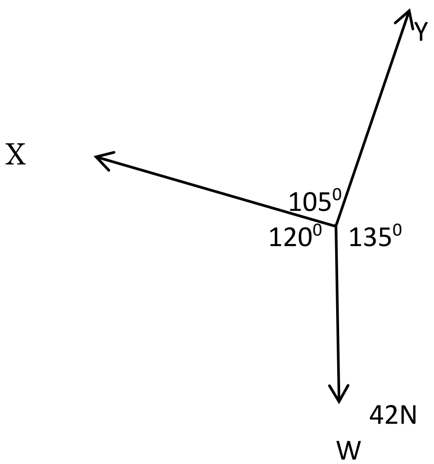
Solution

The bulb is static and hence the three forces T_x, T_y and T_w are in equilibrium.

Hence, the angles at equilibrium can be calculated.



Hence the forces at equilibrium are



From Lami's theory.

$$\frac{T_x}{\sin 135^0} = \frac{T_y}{\sin 120^0} = \frac{T_w}{\sin 105^0}$$

i) $\frac{T_x}{\sin 135^0} = \frac{42}{\sin 105^0}$

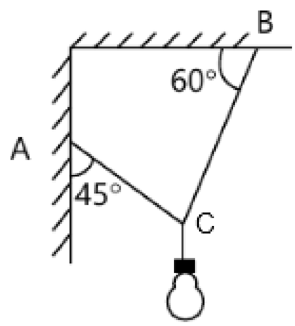
$T_x = \frac{42 \times \sin 135^0}{\sin 105^0} = 30.7$ N to 1d.p.

$$ii)\quad \frac{T_y}{\sin 120^0} = \frac{42}{\sin 105^0}$$

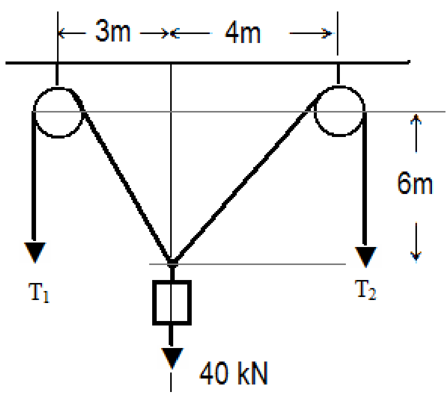
$$T_y = \frac{42 \times \sin 120^0}{\sin 105^0} == 37.7 \text{ N to 1d.p.}$$

Exercise 10.4: Work in pairs.

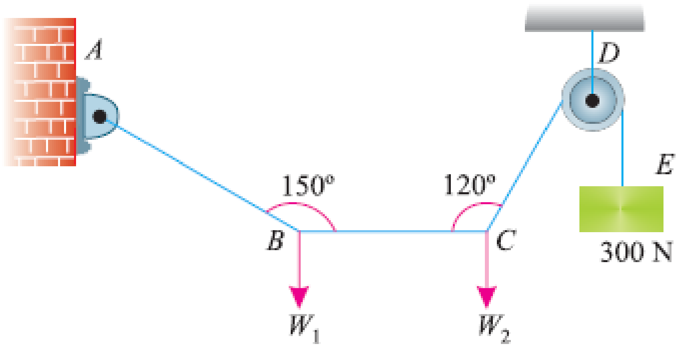
1. A security light bulb weighing 12 N is hung by a string as shown in the figure. Determine the tensions in AC and BC of the string.



2. A partially filled bag of cement of 40N is hung by a string across two friction free pulleys as shown in figure 10.10 below. Calculate the tensions T_1 and T_2 in the string.



3. A smooth sphere of diameter 20 cm weighing 20 N is supported by a string fixed to the wall. The length of the string is three times the radius of the sphere. Determine the reaction at the wall and the tension in the string.
   Hint: Draw a diagram
4. A string ABCDE with one end A fixed to the wall and the other end E loaded with a weight of 300N. The string passes through a frictionless pulley and another two weights are hung at B and C as shown in figure. Determine the tension in the string and the magnitude of weights W_1 and W_2.



Momentum and Impulse

Any moving object can have **momentum**. This is because momentum is mass in motion. The way we determine an object's momentum is fairly straightforward. Momentum is the object's **mass** multiplied by its **velocity**, or, in equation form, $\boldsymbol{p} = \boldsymbol{mv}$, where p is momentum, m is mass in kilograms, and v is velocity in metres per second.

Momentum is proportional to both mass and velocity, meaning that a change in one will cause a proportional change in the other. So if you increase an object's mass, you also increase its momentum. The same is true for velocity: increase or decrease the object's speed, and you increase or decrease its momentum proportionally.

Usually, it is the object's velocity that changes rather than its mass. You may remember that a change in velocity means the object is accelerating. You may also remember that **acceleration** is caused by a force and that the greater the force, the greater the acceleration. Therefore, the greater the acceleration, the greater the momentum,

Force is an important factor, but time also counts. Specifically, we are interested in knowing how long the force acts. For example, if you push a box across the floor for just a few seconds, the time interval is very short. But if you push a box across the floor and you do so with the same force as before, but this time for several minutes, you've increased the amount of time the force acts. This longer time interval leads to a greater change in momentum. This change in momentum is called **impulse**, and it describes the quantity that we just saw: the force multiplied by the time interval it acts for. The greater the impulse, the greater the change in momentum. To change the impulse, you can either change the amount of force, or you can change the time interval during which the force acts. In equation form, we can write this relationship between impulse and momentum as:

$$Ft = \Delta mv$$

The Greek letter *delta* means 'change in,' and we read this equation as force multiplied by the time interval equals the change in momentum (mass multiplied by velocity).

Be careful not to read this as 'force multiplied by time equals mass multiplied by velocity' because that suggests 'impulse equals momentum.' It is important to remember that impulse is a change in momentum, not momentum itself.

Example 1

Calculate the momentum of a rock, 30 kg, rolling at a velocity of 100 m/s.

Solution

Momentum = mass × velocity

$$P = mv$$

$$= 30 \times 100$$

$$= 3000\ k\text{gm/s}$$

Example 2

A rock 5kg experiences a 10N force for a duration of 0.1 second. Calculate the change in momentum of the rock.

Solution

Δ Momentum = Impulse = F.t = 10 × 0.1 = 1 N/s

Exercise 10.5: Work in groups.

1. A football player applies an average force of 80N to 0.25kg ball for 0.10 second. Determine the impulse experienced on the ball.
2. Ann, 50kg is driving her car at a speed of 35 m/s. She notices a pot-hole and suddenly applies the brakes. She is wearing a seat belt and strikes an air bag together they stop her in 0.5 second.
   a) What is the average force exerted by the air bag and seat belt on her?
   b) If there were no seat belt and air bag the wind shield would stop her in 0.02 sec, at what force?
3. Calculate the momentum of a car of 1.5 tonne moving with a speed of 80 km/h.
4. Calculate the force which makes a 12m/s change in velocity of an object with mass 16kg in 4 sec.
5. A force acts on a stone of mass 4kg resulting in a velocity of 8m/s. Assuming the stone was initially at rest, find the impulse of the force.
6. A ball of mass 1kg travelling with velocity 4m/s hits a wall and then returns along the same line of travel with speed 2m/s.
   a) Find the impulse of the wall on the ball.