South Sudan

# Secondary Mathematics 4

Secondary Mathematics has been written and developed by Ministry of General Education and Instruction, Government of South Sudan in conjunction with Subjects experts. This course book provides a fun and practical approach to the subject of mathematics, and at the same time imparting life long skills to the pupils.

The book comprehensively covers the Secondary 4 syllabus as developed by **Ministry of General Education and Instruction.**

**Each year comprises of a Student's Book and Teacher's Guide.**

**The Teacher's Guide provide:**

- Full coverage of the national syllabus.
- A strong grounding in the basics of mathematics.
- Clear presentation and explanation of learning points.
- A wide variety of practice exercises, often showing how mathematics can be applied to real-life situations.
- It provides opportunities for collaboration through group work activities.
- Stimulating illustrations.

**All the courses in this primary series were developed by the Ministry of General Education and Instruction, Republic of South Sudan. The books have been designed to meet the primary school syllabus, and at the same time equiping the pupils with skills to fit in the modern day global society.**




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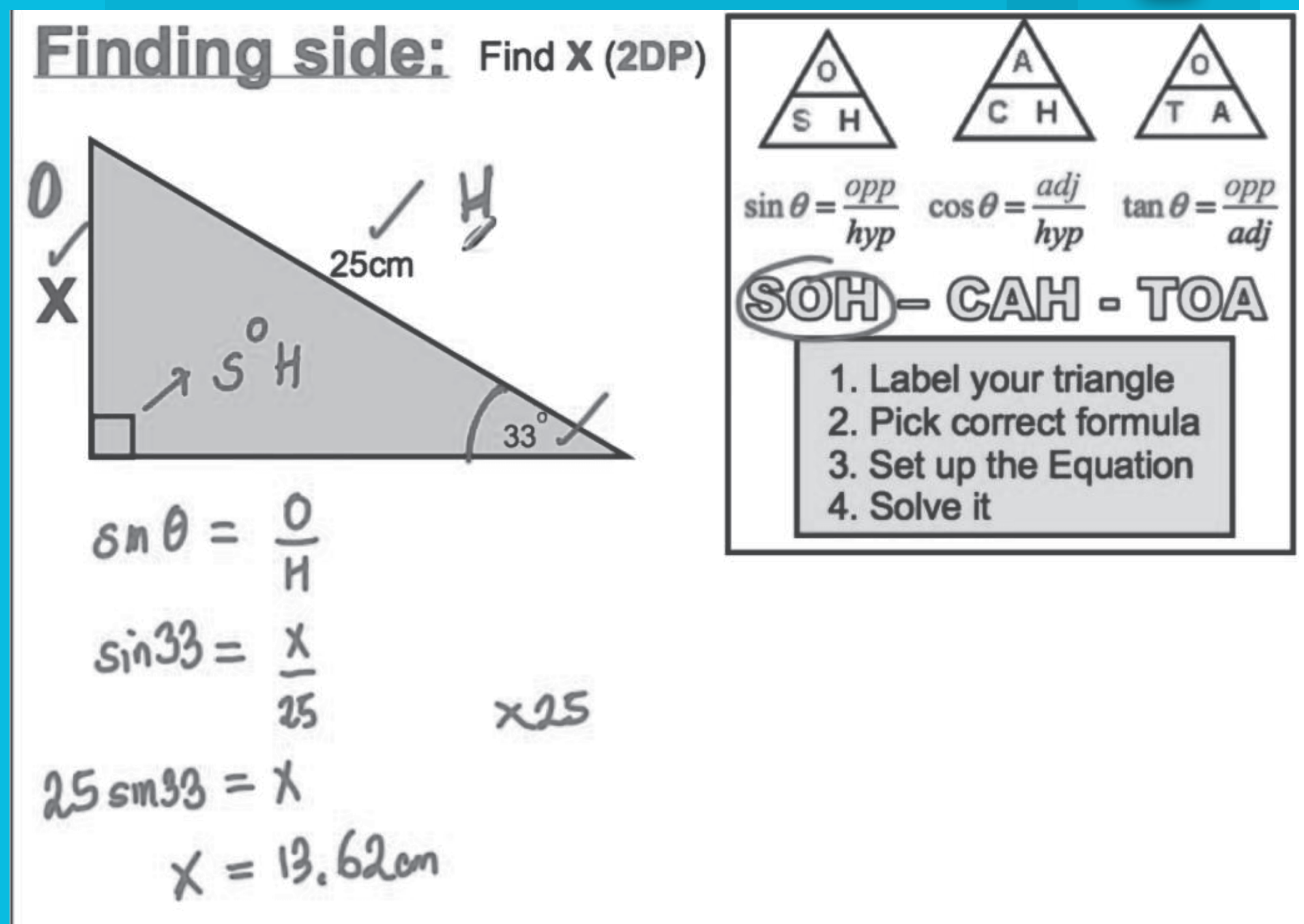
South Sudan


# Secondary Mathematics 4

## Teacher's Guide







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South Sudan

SECONDARY
4

# Mathematics

*Teacher's Guide 4*





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# FOREWORD

I am delighted to present to you this Teacher's Guide, which is developed by the Ministry of General Education and Instruction based on the new South Sudan National Curriculum. The National Curriculum is a learner-centered curriculum that aims to meet the needs and aspirations of the new nation. In particular, it aims to develop (a) Good citizens; (b) successful lifelong learners; (c) creative, active and productive individuals; and (d) Environmentally responsible members of our society. This textbook, like many others, has been designed to contribute to achievement of these noble aims. It has been revised thoroughly by our Subject Panels, is deemed to be fit for the purpose and has been recommended to me for approval. Therefore, I hereby grant my approval. This Teacher's Guide shall be used to facilitate learning for learners in all schools of the Republic of South Sudan, except international schools, with effect from 4th February, 2019.

I am deeply grateful to the staff of the Ministry of General Education and Instruction, especially Mr Michael Lopuke Lotyam Longolio, the Undersecretary of the Ministry, the staff of the Curriculum Development Centre, under the supervision of Mr Omot Okony Olok, the Director General for Quality Assurance and Standards, the Subject Panelists, the Curriculum Foundation (UK), under the able leadership of Dr Brian Male, for providing professional guidance throughout the process of the development of National Curriculum, school textbooks and Teachers' Guides for the Republic of South Sudan since 2013. I wish to thank UNICEF South Sudan for managing the project funded by the Global Partnership in Education so well and funding the development of the National Curriculum, the new textbooks and Teachers' Guides. I am equally grateful for the support provided by Mr Tony Calderbank, the former Country Director of the British Council, South Sudan; Sir Richard Arden, Senior Education Advisor of DfID, South Sudan. I thank Longhorn and Mountain Top publishers in Kenya for working closely with the Ministry, the Subject Panels, UNICEF and the Curriculum Foundation UK to write the new textbooks. Finally, I thank the former Ministers of Education, Hon. Joseph Ukel Abango and Hon. Dr John Gai Nyuot Yoh, for supporting me, in my role as the Undersecretary, to lead the Technical Committee to develop and complete the consultations on the new National Curriculum Framework by 29 November 2013.

The Ministry of General Education and Instruction, Republic of South Sudan, is most grateful to all these key stakeholders for their overwhelming support to the design and development of this historic South Sudan National Curriculum. This historic reform in South Sudan's education system is intended to benefit the people of South Sudan, especially the children and youth and the future generations. It shall enhance the quality of education in the country to promote peace, justice, liberty and prosperity for all. I urge all Teachers to put this textbook to good use.

May God bless South Sudan. May He help our Teachers to inspire, educate and transform the lives of all the children and youth of South Sudan.

Deng Deng Hoc Yai, (Hon.)

Minister of General Education and Instruction, Republic of South Sudan

# Table of Contents

## Introduction

Teaching Mathematics is taking place in rapidly changing conditions. It is necessary to look for optimal didactic and educational solutions encompassing goals and contents as well as forms and teaching methods allowing for preparing students to face the challenges of the contemporary world.

The most significant role of educational system in terms of teaching Mathematics is developing and promoting subject competences as an important factor fostering student's personal development and the development of society. Well organised mathematical education facilitates logical thinking and expressing ideas, organizing own work, planning and organizing the learning process, collaboration and responsibility; it prepares for life in a modern world and enables to perform many jobs.

The teacher is required to pay more attention to students' awareness of developing learning skills and study habits, recognizing and analysing problems and predicting solutions to them. Undeniably, the implementation of modern teaching methods and techniques enhances students' curiosity about Mathematics and increases their understanding of the basis of mathematical and scientific knowledge. In accordance with the trends teaching Mathematics is supposed to help students understand and solve everyday problems.

The aim of teaching Secondary Mathematics is to encourage contemporary students to work in class, acquire knowledge and skills that are necessary in life. Moreover, research shows that teachers applying active methods assess the effectiveness of their work and how students respond to this way of teaching.

## About this guide

The purpose of this guide is to offer suggestions that are helpful to Secondary 4 Mathematics teachers on planning, organizing, executing and evaluating the learning and teaching of mathematics. The suggestions will serve as useful starting points to the teachers who are expected to be dynamic innovative and creative to make the leaning process fit the learners.

The guide is to be used alongside Mathematics Students book for secondary 4. It consists of 4 units, in line with secondary 4 mathematics syllabus.

Each of the unit is structured to contain:

1. Introduction
2. Objectives
3. Teaching/Learning Activities
4. Answers to the exercises given in students secondary 4 book.

In each case, the introduction highlight the relevant work than learners are expected to have covered in their previous mathematics units and what they are expected to have covered previously. It also highlights what they are expected to cover in the unit. The teacher is expected to make a quick lick up of previously learnt concepts. Learners should be able to make relevant references to their previous work. Where possible the mathematics teach ma make an entry behavior evaluation as a revision on previously learnt units related to the unit under study.

The unit objectives specify the skills (cognitive, affective, and psychomotor) that teachers will use to enable learners understand each unit. The objectives are likely to serve a useful purpose if they when stated to reflect the local conditions of the learner. For example, the type of students and the available learning resources. The teacher may break down the unit objectives to various objectives that enhance the learners understanding of the process involved and to suit different situations in the lesson, schools, society and the world at large.

Teaching/learning activities highlight the most noticeable and important. Points encountered in the learning process and suitable techniques to be used in handling each objective(s).

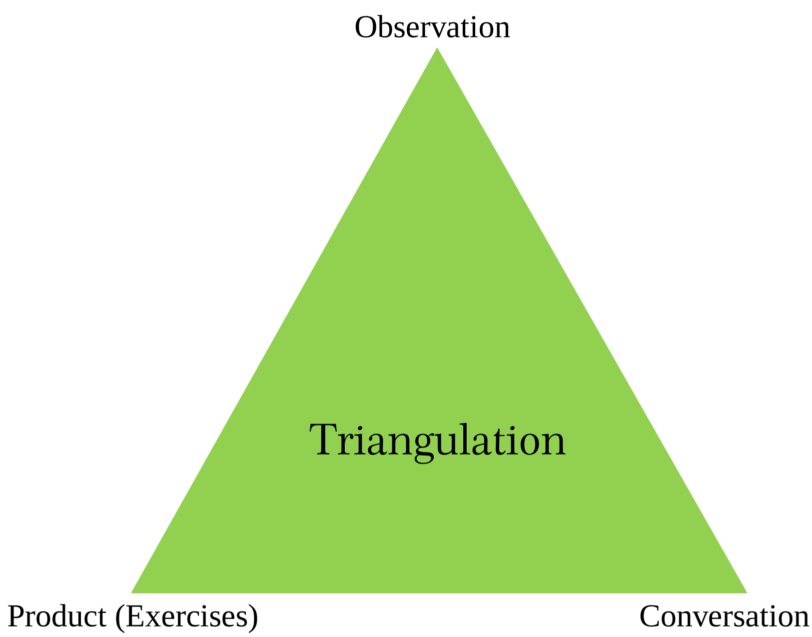
Answers to each exercise in the students' book are provided in these teachers guide. It is contemplated that the most conducive and favorable outcome from the guide will be realized if other sources of learning mathematics are properly organized and used.

Among others, the following should be used alongside the guide:

1. The Schemes of work
2. The teacher's lessons plans.
3. The Records of the work covered by the learners.

## Making Classroom Assessment

- Observation – watching learners as they work to assess the skills learners are developing.
- Conversation – asking questions and talking to learners is good for assessing knowledge and understanding of the learner.
- Product – appraising the learner's work (writing report or finding, mathematics calculation, presentation, drawing diagram, etc.).



To find these opportunities, look at the “Learn About’ sections of the syllabus units. These describe the learning that is expected and in doing so they set out a range of opportunities for the three forms of opportunity.

## Unit 1

# CONJUGATES AND COMPLEX NUMBER

| Math Secondary 4 | Unit 1: Conjugates and Complex Numbers |
|---|---|
| Learn about | Key inquiry questions |
| Learners should revisit the development of a number system through real numbers and know what a complex number is, and the operations on them. They should learn about conjugates, e.g. $a + ib$ is the conjugate of $a - ib$, where a and b are real numbers.<br><br>Learners should investigate complex numbers by putting emphasis on the importance of conjugates as the numerator and the denominator must be multiplied by the conjugate of the denominator. They should represent complex numbers graphically where the X-axis represents the real part and the Y-axis the imaginary part.<br><br>They should investigate the polar form of complex numbers and the relation between polar and Cartesian coordinates where $\cos\theta = x/r$ and $\sin\theta = y/r$ and explain how complex numbers are added, subtracted, multiplied, divided and can be graphically represented on the Cartesian plane. | • What is a complex number?<br>• How do we perform the four operations on complex numbers? How do we represent complex numbers on the Cartesian plane?<br>• What is the polar form of complex numbers? |

| Learning outcomes | | |
|---|---|---|
| Knowledge and understanding | Skills | Attitudes |
| • Understand how complex numbers are added, subtracted, multiplied, divided and can be graphically represented on the Cartesian plane | • Investigate complex numbers by putting emphasis on the importance of conjugates<br>• Represent complex numbers on the Cartesian plane<br>• Apply complex numbers in problem solving<br>• Investigate the polar form of complex numbers | • Appreciate and value complex numbers as a useful tool for solving complex problems |
| Contribution to the competencies:<br><br>Critical thinking through analysis and investigation<br><br>Co-operation and communication through teamwork | | |
| Links to other subjects:<br><br>Physics: Electricity | | |

**Objectives**

By the end of the lesson, the learner should be able to:

i. Add, subtract and multiply and divide complex numbers.
ii. Graphically represent complex numbers on the Cartesian plane
iii. Apply the knowledge on complex numbers in solving problems
iv. Investigate the polar form of complex numbers

**Activities**

- The teacher should guide the learners in discussing the development of a number system through real numbers and to know what a complex number is.
- The teacher should lead the learners to identify conjugates of complex numbers
- The teacher should lead the learners to add and subtract complex numbers using the given examples and any other relevant example.
- The learner should do exercise 1.
- The teacher should lead the learner in multiplication and division of complex numbers.
- The learner should do exercise 2
- The teacher should lead the learners in identifying the real and the imaginary parts of complex numbers and how to represent them on a Cartesian plane
- The learner should do exercise 3
- The teacher should guide the learners on how to investigate the polar form of complex numbers and the relation between polar and the Cartesian coordinates
- Learner should do exercise 4

## History of complex numbers

"*i* is as amazing number. It is the only imaginary number. However, when you square it, it becomes real. Of course, it wasn't instantly created. It took several centuries to convince certain mathematicians to accept this new number.

Eventually, though, a section of numbers called "imaginary" was created (which also includes complex numbers, which are numbers that have both a real and imaginary part), and people now used *i* in everyday math.

*i* was created due to the fact that people simply needed it. At first, solving problems such as "$\sqrt{-39}$" and "$x^2 + 1 = 0$" were thought to be impossible. However, mathematicians soon came up with the idea that such a number to solve these equations could be created.

Today, the number is√-1, more commonly known as *i*. It's a good thing that scientists, mathematicians who didn't want a new numbers created, and other non-believers finally allowed *i* (and complex numbers) in the number system. Today, *i* is very useful to the world.

Engineers use it to study stresses on beams and to study resonance. Complex numbers help us study the flow of fluid around objects, such as water around a pipe. They are used in electric circuits, and help in transmitting radio waves. So, if it weren't for *i*, we might not be able to talk on cell phones, or listen to the radio! Imaginary numbers also help in studying infinite series. Lastly, every polynomial equation has a solution if complex numbers are used. Clearly, it is good that *i* was created.

The very first mention of people trying to use imaginary numbers dates all the way back to the 1stcentury. In 50 A.D., Heron of Alexandria studied the volume of an impossible section of a pyramid. What made it impossible was when he had to take√81-114. However, he deemed this impossible, and soon gave up. For a very long time, no one tried to manipulate imaginary numbers. Although, it wasn't for a lack of trying.

Once negative numbers were "invented", mathematicians tried to find a number that, when squared, could equal a negative one. Not finding an answer, they gave up. In the 1500's, some speculation about square roots of negative numbers was brought back. Formulas for solving 3rd and 4th degree polynomial equations were discovered, and people realized that some work with square roots of negative numbers would occasionally be required. Naturally, they didn't want to work with that, so they usually didn't. Finally, in 1545, the first major work with imaginary numbers occurred.

In 1545, Girolamo Cardano wrote a book titled *Ars Magna*. He solved the equation $\boldsymbol{x(10-x)=40}$, finding the answer to be 5 plus or minus√-15. Although he found that this was the answer, he greatly disliked imaginary numbers. He said that work with them would be, "as subtle as it would be useless", and referred to working with them as "mental torture." For a while, most people agreed with him. Later, in 1637, Rene Descartes came up with the standard form for complex numbers, which is a+b*i*. However, he didn't like complex numbers either. He assumed that if they were involved, you couldn't solve the problem. Lastly, he came up with the term "imaginary", although he meant it to be negative. Issac Newton agreed with Descartes, and Albert Girad even went as far as to call these, "solutions impossible". Although these people didn't enjoy the thought of imaginary numbers, they couldn't stop other mathematicians from believing that *i* might exist.

Rafael Bombelli was a firm believer in complex numbers. He helped introduce them, but since he didn't really know what to do with them, he mostly wasn't believed. He did understand that *i*times *i* should equal -1, and that –*i* times *i* should equal one. Most people did not believe this fact either. Lastly, he did have what people called a "wild idea"- the idea that you could use imaginary numbers to get the real answers. Today, this is known as conjugation. Although Bombelli himself did not have much of an impact at the time, he helped lead the way for imaginary numbers.

Over decades, many people believed that complex numbers existed, and set out to make them understood and accepted. One of the ways they wanted to make them accepted was to be able to plot them of a graph. In this case, the *x*-axis is would be real numbers, and the Y-axis would be imaginary numbers. If the number were purely imaginary (like 2*i*), it would just be on the Y-axis. If the number was purely real, it would just be on the X-axis. The first person who considered this kind of graph was John Wallis. In 1685, he said that a complex number was just a point on a plane, but he was ignored. More than a century later, Caspar Wessel published a paper showing how to represent complex numbers in a plane, but was also ignored. In 1777, Euler made the symbol *i* stand for √-1, which made it a little easier to understand. In 1804, Abbe Buee thought about John Wallis's idea about graphing imaginary numbers, and agreed with him. In 1806, Jean Robert Argand wrote how to plot them in a plane, and today the plane is called the Argand diagram. In 1831, Carl Friedrich Gauss made Argand's idea popular, and introduced it to many people. In addition, Gauss took Descartes' a + b*i* notation, and called this a complex number. It took all these people working together to get the world, for the most part, to accept complex numbers.

Mathematicians kept working to make sure that imaginary and complex numbers were understood. In 1833, William Rowan Hamilton expressed complex numbers as pairs of real numbers (such as 4+3*i* being expresses as (4,3)), making them less confusing and even more believable. After this, many people, such as Karl Weierstrass, Hermann Schwarz, Richard Dedekind, Otto Holder, Henri Poincare, Eduard Study, and Sir Frank Macfarlane Burnet all studied the general theory of complex numbers. Augustin Louis Cauchy and Niels Henrik Able made a general theory about complex numbers accepted. August Mobius made many notes about how to apply complex numbers in geometry. All of these mathematicians helped the world better understand complex numbers, and how they are useful.

Clearly, complex numbers are amazing. They have many uses, more than we realize. They have a fascinating history, full of some mathematicians not believing in them and others desperately trying to prove their existence. *i* is also fascinating, being the only imaginary number. Many mathematicians brought together as much proof as they could that imaginary numbers should exist, and we have them to thank today that we can use *i* whenever we please, without being questioned about it."

**An alternate answer without History to why imaginary numbers were invented.**

In a number system, we want the system to be closed. This means if we do a mathematical operation on any number(s), we get a number back inside the number system.

For example, integers are closed over addition. 1 + -2 = -1.
(You can always add two integers and get another integer.)
By they are not closed over division 1 / 2 = 0.5, not an integer.

The most famous example, of course, is the square root of negative 1: this is does belong to "Real numbers" which includes any number that can be represented as a decimal (either finite or infinite).

This presented a need to create numbers that allowed us to do calculations using numbers such as the square root of one. I know, for example, this is very common in physics and thus engineering. And since need is the mother of invention, complex numbers were born to create a truly closed number system.

## Exercise 1

The teacher to guide learners in solving the problems presented at the start of the unit. The teacher to allow learners to share their observations and answers after completing the exercise. The teacher should make sure that the learners in pairs, communicate and work collaboratively.

In pairs, work out the following problems.

1. Explain why $\sqrt{-16}$ is not a real number.
2. Add: $(2x + 5) + (3x + 7)$
3. Subtract: $(x - 1) - (4x + 8)$
4. Multiply: $9x(5x + 6)$
5. Multiply: $(2x + 3)(7x - 1)$
6. Solve: $x^2 - 24 = 0$

Some quadratic equations do not have real-number solutions. When we use the square root property or the quadratic formula to solve them, we encountered the square root of a complex number.

## Representation of complex numbers on the Argand diagram.

The teacher will guide the learners to:

1. represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
2. represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

## Exercise 2

Teacher to guide learners in completing the exercise.

1. Plot the following point on the Argand diagram.
   i) A = 2 + i
   ii) B = 2 + 3i
   iii) C = -i + 4
   iv) D = ½ + -i
2. What is the mid-point of A = 1+8i and B = -5 +3i?
3. Consider two points A = -4 + 5i and B =4 -10i,
   a) Find the mid-point of A and B.
   b) Find the distance between A and B.

***Solutions***

1. Plot A on $(\mathbf{2},\mathbf{1})$, plot B on $(\mathbf{2},\mathbf{3})$, plot C on $(\mathbf{4},-\mathbf{1})$, plot D on $(-\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$
2. $(-\mathbf{2}+\frac{\mathbf{5}}{\mathbf{2}}\boldsymbol{i})$
3. a)$-\frac{\mathbf{5}}{\mathbf{2}}\boldsymbol{i}$

   b)$\mathbf{17}\ \boldsymbol{units}$

## Operations on complex numbers

The teacher should ensure each learner knows that there is a complex number $\boldsymbol{i}$ such that $\boldsymbol{i}^2 = -1$, and every complex number has the form $\boldsymbol{a} + \boldsymbol{bi}$ with $\boldsymbol{a}$ and $\boldsymbol{b}$ real. Use the relation $\boldsymbol{i}^2 = -\mathbf{1}$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

1. Find the conjugate of a complex number; use conjugates to find the moduli and quotients of complex numbers.

**Exercise 3**

1. Work out the following
   a) $(-\mathbf{4} + \mathbf{7}\boldsymbol{i}) + (\mathbf{5} - \mathbf{10}\boldsymbol{i})$
   b) $(\mathbf{4} + \mathbf{12}\boldsymbol{i}) - (\mathbf{3} - \mathbf{15}\boldsymbol{i})$
   c) $\mathbf{5}\boldsymbol{i} - (-\mathbf{9} + \boldsymbol{i})$
   d) $(\mathbf{3} + \mathbf{4}\boldsymbol{i}) - (\mathbf{6} - \mathbf{10}\boldsymbol{i})$
   e) $\mathbf{2}\,(-\mathbf{3}\boldsymbol{i} + \mathbf{4}\boldsymbol{i}) - \mathbf{3}\,(\mathbf{5} - \boldsymbol{i})$

***Solutions***

a) $-\mathbf{1} - \mathbf{3}\boldsymbol{i}$
b) $\mathbf{1} + \mathbf{27}\boldsymbol{i}$
c) $\mathbf{9} + \mathbf{4}\boldsymbol{i}$
d) $-\mathbf{3} + \mathbf{14}\boldsymbol{i}$

e) $\boldsymbol{-21 + 11i}$

**Multiplying complex numbers**
**Task**

Multiply and simplify $\boldsymbol{(6 + 8i)(6 - 8i)}$

What do you notice?

| Problem | Multiply and simplify $(6 + 8i)(6 - 8i)$ | |
|---|---|---|
| | $(6 + 8i)(6 - 8i)$<br>$= 6(6) + 6(-8i) + 8i(6) + 8i(-8i)$<br>$= 36 - 48i + 48i - 64i^2$ | Expand the product. |
| | $= 36 - 64i^2$ | Combine like terms. |
| | $= 36 - 64(-1)$<br>$= 36 + 64$<br>$= 100$ | Replace $i^2$ with $-1$ and simplify. |
| *Answer* | $(6 + 8i)(6 - 8i) = 100$ | |

Just as $\boldsymbol{6 + 8\sqrt{3}}$ and $\boldsymbol{6 - 8\sqrt{3}}$ are *conjugates*, $6 + 8i$ and $6 - 8i$ are conjugates. (Again, $i$ is a square root, so this isn't really a new idea.) When the numbers are complex, they are called complex conjugates. Because conjugates have terms that are the same except for the operation between them (one is addition and one is subtraction), the $i$ terms in the product will add to 0. In the example above, $-48i$ was added to $48i$, and that sum is 0, so there was no $i$ term in the final product. That means the product of complex conjugates will always be a real (not complex) number.

**Exercise 4: Work in pairs.**

**Simplify.**

1.
2. $(-4\boldsymbol{i})^3$
3. $(\boldsymbol{i})(2\boldsymbol{i})(-7\boldsymbol{i})$
4. $(3\boldsymbol{i})(-2+6\boldsymbol{i})$
5. $(-2\boldsymbol{i})^3$
6. $(6\boldsymbol{i})(-4\boldsymbol{i})$
7. $(-8\boldsymbol{i})^2$
8. $(-3+5\boldsymbol{i})(-5+7\boldsymbol{i})$
9. $(5-3\boldsymbol{i})(-8+5\boldsymbol{i})$
10. $(5+5\boldsymbol{i})(-3-7\boldsymbol{i})$
11. $(-7-4\boldsymbol{i})(-6-6\boldsymbol{i})$

***Solutions***

1. $64\boldsymbol{i}$
2. $12\boldsymbol{i}$
3. $-8-6\boldsymbol{i}$
4. $8\boldsymbol{i}$
5. $24$
6. $-64$
7. $-2—46\boldsymbol{i}$
8. $-25+49\boldsymbol{i}$
9. $20-50\boldsymbol{i}$
10. $18+66\boldsymbol{i}$

**Dividing complex numbers**
**Solutions to Exercise 5**

1. Simplify the following complex numbers.
   a) $(4 + 3\boldsymbol{i})\,(2 - \boldsymbol{i})$
   b) $\frac{1+2i}{3-4i}$
   c) $7\boldsymbol{i}(-5+2\boldsymbol{i})$
   d) $(1-5\boldsymbol{i})\,(-9+2\boldsymbol{i})$
   e) $(1-8\boldsymbol{i})(1+8\boldsymbol{i})$
   f) $(5+\boldsymbol{i})\,2(+6\boldsymbol{i})$
2. Verify the following.

a) $(\sqrt{2} - i)$ - i (1 - $\sqrt{2}i$) =-2i

***Solutions***

a) $\mathbf{14 + 2i}$
b) $\frac{1}{2}i - 5$
c) $-14 - 35i$
d) $1 - 41i$
e) $65$
f) $4 + 32i$

2.

a) $-2i$
b) $-\frac{2}{5}$
c) $\frac{5}{-10i}$
d) $-4$

## Polar form of complex numbers

**Solutions to Exercise 6**

1. a) $5.39$

b) $z_1 = 3(\cos 90^0 + i \sin 90^0)$

$$z_2 = 2.8\ (\cos -33.7^0 + i \sin -33.7^0)$$

2. i) $z = 10(\cos 36.87^0 + i \sin 36.87^0)$
ii) $z = 5(\cos -53.13^0 + i \sin -53.13^0)$
iii) $z = 0.7036(\cos 185.82^0 + i \sin 185.82^0)$

3. a) $z = 3.54 - 3.54i$
b) $z = 5.2 - 3i$
c) $z = 0.41 + 0.29i$

4. $p(3,\ 6.3i)$

## Exercise 7

1. Add and express in the form of a complex number $\boldsymbol{a + b\,i}$.
$$(2 + 3i) + (-4 + 5i) - (9 - 3i) / 3$$

2. Multiply and express in the form of a complex number $\boldsymbol{a + b\,i}$.
$$(-5 + 3i)(-4 + 8i)$$

3. Divide and express in the form of a complex number a + b i.
$$(-1 - 2i) / (-4 + 3i)$$

4. Find the complex conjugate to.
$$1 + 8i$$

5. Express in the form of a complex number $\boldsymbol{a + b\,i}$.
$$(-5 - i)(-7 + 8i) / (2 - 4i)$$

6. Express in the form of a complex number $\boldsymbol{a + b\,i}$.
$$-(7 - i)(-4 - 2i)(2 - i)$$

7. Express in the form of a complex number $\boldsymbol{a + b\,i}$.
$$i / (1 - i)$$

8. Solve for x and y where x and y are real numbers.
$$2y + ix = 4 + x - i$$

9. Find a and b, where a and b are real numbers so that
$$a + ib = (2 - i)2$$

10. Find the complex conjugate to $\boldsymbol{-3\,i}$.

***Solutions***

**question 1:** -5 + 9 i

**question 2:** -4 - 52 i

**question 3:** -2 / 25 + (11 / 25) i

**question 4:** 1 - 8 i

**question 5:** 109 / 10 + (53 / 10) i

**question 6:** 70 - 10 i

**question 7:** -1 / 2 + i / 2

**question 8:** x = -1 , y = 3 / 2

**question 9:** a = 3 , b = -4

**question 10:** 3 i

## Exercise 8: Application Questions

1. In an AC (alternating current) circuit, if two sections are connected in series and have the same current in each section, the voltage is given by $\boldsymbol{V = V_1 + V_2}$. Find the total voltage in a given circuit if the voltages in the individual sections are $\boldsymbol{V_1 = 10.31 - 5.97i}$ and $\boldsymbol{V_2 = 8.14 + 3.79i}$.

2. The impedance $\boldsymbol{Z}$ in an AC (alternating current) circuit is a measure of how much the circuit impedes (hinders) the flow of current through it. The impedance is related to the voltage $\boldsymbol{V}$ and the current $\boldsymbol{I}$ by the formula$\boldsymbol{V = IZ}$.

   If a circuit has a current of $(\boldsymbol{0.5 + 2.0i})$ amps and an impedance of $(\boldsymbol{0.4 - 3.0i})$ ohms, find the voltage.

**Real Life Context Complex**

Numbers are useful in representing a phenomenon that has two parts varying at the same time, for example an alternating current. Also, radio waves, sound waves and microwaves have to travel through different media to get to their final destination. There are many instances where, for example, engineers, doctors, scientists, vehicle designers and others who use electromagnetic signals need to know how strong a signal is when it reaches its destination. The two parts in this context are: the rotation of the signal and its strength. The following are examples of this phenomenon:

- A microphone signal passing through an amplifier
- A mobile phone signal travelling from the mast to a phone a couple of miles away
- A sound wave passing through the bones in the ear
- An ultrasound signal reflected from a foetus in the womb
- The song of a whale passing through miles of ocean water

Complex Numbers are also used in:

- The prediction of eclipses
- Computer game design
- Computer generated images in the film industry
- The resonance of structures (bridges, etc.)
- Analysing the flow of air around the wings of a plane in aircraft design

**UNIT 2**

# MEASUREMENT & TRIGONOMETRY

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<tr><td>Maths Secondary 4</td><td>Unit 2: Measurement and Trigonometry</td></tr>
<tr><td>Learn about</td><td>Key inquiry questions</td></tr>
<tr><td>Learners should know how to approximate areas of irregular objects by the methods of :-<br><br>a) Counting the squares on grid paper covering an irregular area.<br>b) Using trapezium rule and mid-ordinate rule.<br>c) Explain how to use grid paper to find area of irregular objects.<br>d) Describe how to approximate an irregular area by using trapezium and mid-ordinate rules.<br>e) Explain and carry out the above techniques to approximate the area of South Sudan using its scale drawn map.<br><br>Learners should investigate the properties of a circle passing through two points and touching the x-axis and develop the equation<br><br>Learners should investigate how to plot graphs of trigonometric ratios in Trigonometry (lll) and explain how a graph of a trigonometric ratio (sin, cosine and tan) is plotted and know the rules for solving trigonometric problems.</td><td>• What is the area of an irregular object?<br>• How do you use grid paper to approximate an irregular area?<br>• What is the trapezium rule and what is the mid-ordinate rule and how do you find area using either of the two rules?<br>• How do we find the equation of a circle passing through two points and touching the x-axis?<br>• How do we plot the graph of a trigonometric ratio (sine, cosine and tan)?</td></tr>
</table>

| Learning outcomes | | |
|---|---|---|
| Knowledge and understanding | Skills | Attitudes |
| • Understand the trapezium and mid-ordinate rules<br>• Understand the equation of a circle passing through two points touching x-axis<br>• Understand how trigonometrical ratios are plotted | • Use a grid to approximate the area of irregular shapes<br>• Apply trapezium and mid-ordinate rules to approximate irregular areas<br>• Use two points through which a circle is passing to find its equation<br>• Plot graphs of trigonometric ratios | • Appreciate, and value equations in plotting trigonometric graphs |
| Contribution to the competencies:<br><br>Communication in using a range of rules and objects to approximate irregular areas<br><br>Creative thinking and problem solving in finding the equation of a circle and plotting trigonometric ratios | | |
| Links to other subjects:<br><br>Geography: approximating area of a map<br><br>Physics: approximating area of irregular objects; calculating vector forces | | |

**Objectives**

By the end of the topic, the learner should be able to:

- Use a grid to estimate the area of irregular shapes
- Apply trapezium and mid-ordinate rules to approximate irregular area
- Use two points through which a circle is passing to find its equation
- Plot the graphs of trigonometric ratios

**Irregular area**
**Activities**

The learners should use a tracing paper to draw the map of Southern Sudan and other irregular shapes in their exercise books and be guided by the teacher to estimate its area using appropriate grid.

## How to use a grid paper to approximate an irregular area

**The teacher to guide learners in completing the following activities.**

Trace your hand on a piece of paper. Think about how you might determine the area of your handprint.

### Activity 1

Will the amount of area covered differ if you trace your hand with your fingers close together or spread apart? Explain.

Units for measuring area must have the following properties:

- The unit itself must be the interior of a simple closed shape.
- The unit, when repeated, must completely cover the object of interest, with no holes or gaps (like a tessellation). Many polygons (e.g., rectangles, rhombuses, and trapezoids) and irregular shapes (e.g., L shapes) have this property and can thus be used as units of measurement.

### Activity 2

1. What units might you use to determine the area of your handprint?
2. Why is a small circle not suitable as the unit of measurement?

**Activity 3**

1. One method for finding the area of an irregular shape is to count unit squares. Use centimeter grid paper to determine the area of your handprint. What are the disadvantages of this method?
2. Another method is to subdivide your handprint into sections for which you can easily calculate the area. Find the area of your handprint using this method. Does using the two methods result in the same area?

Up until now, you have been approximating the area of your handprint. In other words, your measurements were not exact.

**Activity 4**

What can you do to make your approximation more accurate? Explain why this approach will lead to a better approximation. Another way to approximate the area of a handprint or any other irregular shape is to determine the number of squares that are completely covered and the number of squares that are partially covered. Average these two numbers to get an approximate area in the number of square units.

**Activity 5**

Think about the following statement: If you repeatedly use a smaller and smaller unit to calculate the area of an irregular shape, you will get a closer and closer approximation and eventually find the exact area. What do you think of this line of reasoning? Explain.

**Activity 6**

The palm of your hand is about one percent of your body's surface area. Doctors sometimes use this piece of information to estimate the percent of the body that is affected in burn victims. Use your data to approximate the amount of skin on your body.

**Activity 7: Work in pairs**

**Given the irregular shape;**

1. Sub-divide the area into squares of a unit length.
2. Mark all the whole squares within the area.
3. Mark all incomplete squares with a different mark (e.g. xx).
4. Count the complete squares.
5. Count the incomplete squares.
6. The area is therefore approximated by;
   i. Area = number of complete squares + $\frac{1}{2}$ (number of incomplete squares)

**Exercise 1**

***Solutions***

1. $\mathbf{30\ square\ units}$
2. a) $\mathbf{1.06 \times 10^6 cm}$
   b) $\mathbf{10600\ hectares}$

## Area estimation by using mid- ordinate rule

**Activities**

i. The teacher should guide the learner in deriving the mid- ordinate rule as is in the students' book
ii. The teachers should discuss with learners how to use the mid-ordinate rule to estimate the area of irregular shapes.
iii. The learners should do exercise 7

**Evaluation:** Give a written test on area approximation.

**Solutions to Exercise 2**

1. i) $\mathbf{53.75}\boldsymbol{units}^{\mathbf{2}}$

   ii) $\mathbf{442.75}\boldsymbol{units}^{\mathbf{2}}$

   iii) $\mathbf{1552.712}\boldsymbol{units}^{\mathbf{2}}$
2. $\mathbf{1000}\,\boldsymbol{m}^{\mathbf{2}}$
3.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| y | 10 | 14.67 | 18.67 | 22 | 24.67 | 26.67 | 28 |

Draw the graph

Area =$\mathbf{126.18}\boldsymbol{units}^{\mathbf{2}}$

4. $\boldsymbol{x} = -\mathbf{2}$ **or** $\boldsymbol{x} = \mathbf{6}$

## Area estimation using a trapezium rule.

**Activities**

i. The teacher should guide the learner in deriving the trapezium rule as is in the students' book
ii. The teachers should discuss with learners how to use the trapezium rule to estimate the area of irregular shapes.
iii. The learners should do exercise 6

## Exercise 3

***Solutions***

1. a) $\mathbf{48}\boldsymbol{units}^{\mathbf{2}}$

b)$\mathbf{138.75}\boldsymbol{units}^{\mathbf{2}}$

c)$\mathbf{44.5}\boldsymbol{units}^{\mathbf{2}}$

2. a)

| $x$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 4 | 3.75 | 5 | 7.75 | 8 | 17.75 | 25 | 33.75 | 44 |

b) Draw the graph

c)$\mathbf{125} \boldsymbol{units}^{\mathbf{2}}$

3. $\mathbf{30.8}\boldsymbol{m}$
4. $\mathbf{409.5}$

**Task**

Use the equation $\boldsymbol{y} = \mathbf{3}\boldsymbol{x}^{\mathbf{2}} - \mathbf{2}\boldsymbol{x} + \mathbf{4}$ to complete the table below.

| $x$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | 5 | | | | | 33.75 | |

a) Use integration to find the exact the area bounded by *x*- axis, *y*- axis and the line $\boldsymbol{x} = \mathbf{4}$

b) Use the mid-ordinate rule to estimate the area bounded by *x*- axis, *y*- axis and the line $\boldsymbol{x} = \mathbf{4}$ with 8 rectangles.

c) Use trapezium rule to estimate the area bounded by *x*- axis, *y*- axis and the line $\boldsymbol{x} = \mathbf{4}$ with 8 trapezia.

d) Comment on your different results.

## Equating of a circle

**Activities**

i. The teacher should guide the learner on how to formulate the equation of a circle
ii. The teacher should use the examples in the students book to guide the learner on how to form the equation of a circle given center and radius
iii. The learner should do exercise 8

**Evaluation:** Give a written test on equation of a circle

### Exercise 4

***Solutions***

1. a) $\boldsymbol{x^2 + y^2 - 5 = 0}$

b)$\boldsymbol{x^2 - 4x + y^2 - 2y = 21}$

c)$\boldsymbol{x^2 - 8x + y^2 - 2y - 15 = 0}$

d)$\boldsymbol{x^2 - 8x + y^2 - 10y + 31 = 0}$
e)$\boldsymbol{x^2 - 2x + y^2 + 4y - 24 = 0}$

2. Form the equation of the circle with centre $(\boldsymbol{-3},\ \boldsymbol{1})$ and radius 4 units

3 a) Centre $(\boldsymbol{2}, \boldsymbol{3})$, r = 5

b) Centre (-1, 0), r = 4
b) Centre (0, 0), r =7
d) Centre (2, 3), **r** $= \sqrt{\boldsymbol{17}}$

**The equation of a circle passing through 2 points and touching the x-axis**
**Activities**

i. The learner should be guided through the examples in the book on how to form the equation of a circle touching the x-axis and given 2 points
ii. The learner should do exercise 9.

**Evaluation**: Give a written test on equation of a circle

**Exercise 5**

***Solutions***

1 a) $(\boldsymbol{x}-\mathbf{5.08})^{\mathbf{2}}+(\boldsymbol{y}-\mathbf{5.24})^{\mathbf{2}}=\mathbf{27.46}$

Or $(\boldsymbol{x}-\mathbf{3.74})^{\mathbf{2}}+(\boldsymbol{y}-\mathbf{16.97})^{\mathbf{2}}=\mathbf{287.98}$

b) $(\boldsymbol{x}+\mathbf{4})^{\mathbf{2}}+(\boldsymbol{y}-\mathbf{17})^{\mathbf{2}}=\mathbf{289}$

or$(\boldsymbol{x}-\mathbf{8})^{\mathbf{2}}+(\boldsymbol{y}-\mathbf{37})^{\mathbf{2}}=\mathbf{1369}$

c) $(\boldsymbol{x}-\mathbf{11.6})^{\mathbf{2}}+(\boldsymbol{y}-\mathbf{30.22})^{\mathbf{2}}=\mathbf{913.25}$

or$(\boldsymbol{x}-\mathbf{1.6})^{\mathbf{2}}+(\boldsymbol{y}-\mathbf{10.18})^{\mathbf{2}}=\mathbf{103.63}$

d) $(\boldsymbol{x}-\mathbf{4})^{\mathbf{2}}+(\boldsymbol{y}-\mathbf{5.89})^{\mathbf{2}}=\mathbf{34.69}$

2. i) h = 4, k=3, centre (4, 3)

ii) r= 4 units

iii) $(\boldsymbol{x}-\mathbf{4})^{\mathbf{2}}+(\boldsymbol{y}-\mathbf{3})^{\mathbf{2}}=\mathbf{9}$

## Graphs of trigonometric functions

### Plotting graphs of simple trigonometric functions

**Activities**

i. The learners should be guided on how to plot the graphs of trigonometric functions
ii. The learners should do exercise 10

**Evaluation**: give a written test graphs of trigonometric functions.

**Exercise 6**

*Solutions*

1 a)$\boldsymbol{x = 30^0, x = 300^0}$

b)$\boldsymbol{x = 126^0, x = 232^0}$
c)$\boldsymbol{x = 45^0, x = 318^0}$
d)$\boldsymbol{x = 156^0, x = 204^0}$

2. a) $\boldsymbol{\cos 24^0 = 0.9}$

b)$\boldsymbol{\cos 56^0 = 0.57}$
c)$\boldsymbol{\cos 112^0 = -0.37}$
d)$\boldsymbol{\cos 260^0 = -0.15}$
e)$\boldsymbol{\cos 294^0 = 0.4}$
f)$\boldsymbol{\cos 318^0 = 0.75}$

3. Draw graphs

**UNIT 3**

# INEQUALITIES AND VECTORS

<table>
<tr><td colspan="2">Maths Secondary 4</td><td>Unit 3: Inequalities and Vectors</td></tr>
<tr><td colspan="2">Learn about</td><td>Key inquiry questions</td></tr>
<tr><td colspan="2">Learners should know how to form inequalities and find maximum and minimum values of linear inequalities (use computer linear programming to verify maximum and minimum values of linear inequalities) and know how to apply linear programming.<br><br>Learners should define permutation and combination and know the meaning of factorial (n!) and their application.<br><br>Learners should know about vectors in two- and three-dimensional systems and represent column and position vectors in the three-dimension system.</td><td>• How can we explain the formation of inequalities?<br>• Can you find maximum and minimum values of linear inequalities and apply them in linear programming?<br>• What is a permutation and what is a combination?<br>• What do you know about presentation of vectors in two- and three-dimensional systems?</td></tr>
<tr><td colspan="3">Learning outcomes</td></tr>
<tr><td>Knowledge and understanding</td><td>Skills</td><td>Attitudes</td></tr>
</table>

**Objectives**

<table>
<tr>
<td>Understand:<br><br>• Forming inequalities, maximum and minimum values of linear inequalities and linear programming<br>• Permutation and combination<br>• Ways of arrangement of objects, factorial notation and its application<br>Vectors (III):<br><br>• Understand coordinates in two and three dimension systems, column and position vectors in three dimensions</td>
<td>• Form inequalities<br>• Apply maximum and minimum values of linear inequalities in linear programming and use of computer<br>• Represent vectors in two- and three-dimensional systems<br>• Solve problems using permutation and combination</td>
<td>• Appreciate and value application of inequalities in linear programming<br>• Self-confident with the application of combination and permutation in problem solving</td>
</tr>
<tr>
<td colspan="3">Contribution to the competencies:<br><br>Critical thinking through analysis and investigation<br><br>Co-operation and communication through teamwork</td>
</tr>
<tr>
<td colspan="3">Links to other subjects:<br><br>Physical Education: permutation and combination<br><br>Library Sciences: arranging books in shelves<br><br>Physics: vectors<br><br>ICT: linear programming</td>
</tr>
</table>

By the end of the topic, the learner should be able to:

i. Form linear inequalities based on real life situations
ii. Represent linear inequalities on a graph.

iii. Find maximum and minimum values of linear inequalities
iv. Represent vectors in 2 and 3 dimensions
v. Solve problems using permutations and combinations

## Linear Inequalities

**Activities**

- The teacher should introduce to formation of linear inequalities
- The learner should do exercise 10

**Teacher Notes**

The techniques for solving systems of linear inequalities differ from those for linear equations because the inequality signs do not allow us to perform substitution as we do with equations. Nevertheless, we can still solve these problems.

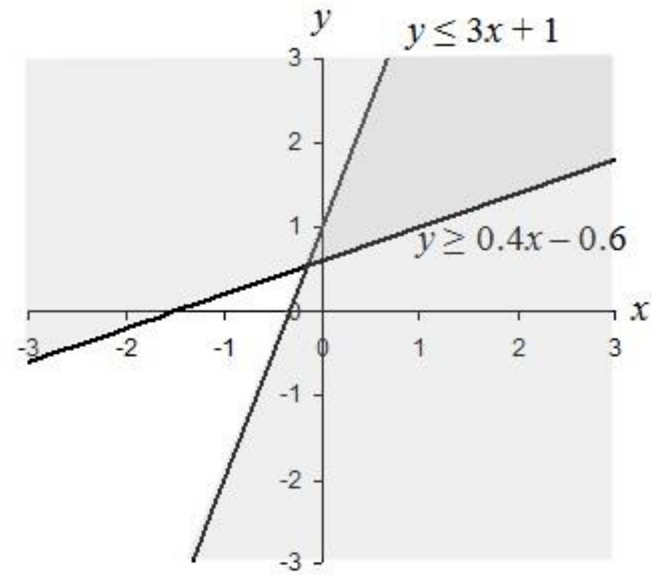
A system of linear inequalities involves several expressions that, when solved, may yield a range of solutions. Many of the concepts we learned when studying systems of linear equations translate to solving a system of linear inequalities, but the process can be somewhat difficult. Perhaps the most lucid way to simultaneously solve a set of linear inequalities is through the use of graphs. Let's consider an example in two dimensions right away.

$$\mathbf{2x - 5y \leq 3}$$
$$\mathbf{y - 3x \leq 1}$$

Because of the inequality, we cannot use substitution in the same way as we did with systems of linear equations. Let's take a look at the graphs of these inequalities. First, we simplify into a form that's easy to plot graphically.

$$\mathbf{2x - 5y \leq 3} \qquad \mathbf{y - 3x \leq 1}$$
$$\mathbf{2x \leq 3 + 5y} \qquad \mathbf{y \leq 3x + 1}$$
$$\mathbf{5y \geq 2x - 3}$$
$$\mathbf{y \geq 0.4\,x - 0.6}$$

Now, we plot the graph of these inequalities.




We can see in the graph that there are two shaded regions corresponding to the solutions to each inequality. The lines are shaded because the inequalities are not strict (≥ and ≤ are used). The solution to the system of inequalities is the darker shaded region, which is the overlap of the two individual regions, and the portions of the lines (rays) that border the region. Symbolically, we can perhaps best express the solution in this case as

$$\mathbf{0.4x - 0.6 \le y \le 3x + 1}$$

Solving systems of inequalities in three or more dimensions is possible, but it is much more complicated-graphing the solid regions that constitute the solutions is likewise tougher.

***Example***

Find and graph the solution set of the following system of inequalities:

$\mathbf{x - 5y \ge 6}$

$\mathbf{3x + 2y > 1}$

***Solution***

First, let's solve the expressions for $y$.

$x - 5y \geq 6$ $\quad\quad$ $3x + 2y > 1$

$x \geq 6 + 5y$ $\quad\quad$ $2y > 1 - 3x$

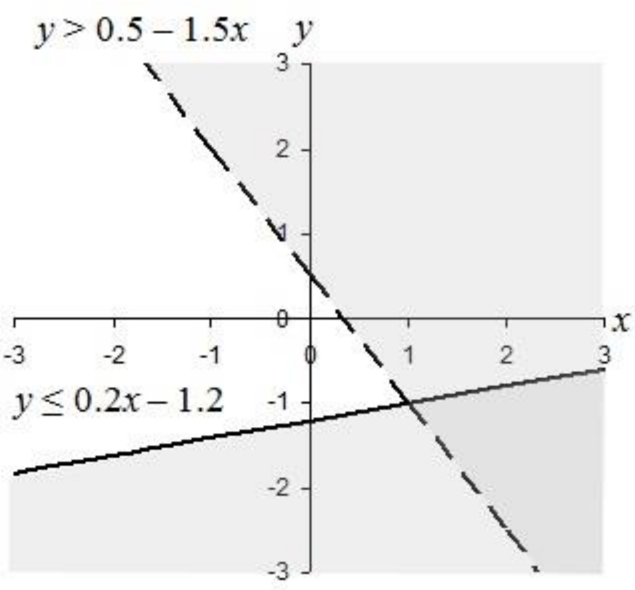
$5y \leq x - 6$ $\quad\quad$ $y > 0.5 - 1.5x$

$y \leq 0.2x - 1.2$

We can then express the solution to this system of inequalities as follows:

$$0.5 - 1.5x < y \leq 0.2x - 1.2$$

Let's graph the solution set. First, we'll graph the lines corresponding to the two individual inequalities (and choosing a solid line for the first and a broken line for the second), then we'll shade the two regions appropriately.




The solution is the darker shaded area (which is the overlap of the two individual solution regions), but let's graph it alone to be a little more clear.

**Exercise 1**

***Solutions***

1. Draw the graphs clearly showing the region R

2 a)$y > x - 2, y \leq 4,\ y \geq 4 - x$

b) $x < 6, \; 2y \leq x + 4, \; y \leq x - 6$

**Five Areas of Application for Linear Programming Techniques**

**Railroads**

Some railroad companies that also own freight train carriages use linear programming techniques to decide how many carriages to store at a particular location. This is so the supply of carriages matches the demand.

**Agriculture**

The classic example of the use of linear programming is in agriculture. Here the thing to be maximized is usually profit and the inputs are constraints like the cost of fertilizer for different crops, the amount of land available, the profit margin per unit of a particular crop, and the amount of a particular crop that can be grown per area of land.

**Warfare**

Linear programming was originally developed during World War II to plan spending on military activities, so as to reduce the army's costs and increase losses for the enemy. Linear programming remains one of many operational research techniques used by armed forces worldwide.

**Telecommunications**

Another application of linear algebra lies in telecommunications. If there are many telephone calls being transmitted across a multipoint phone line network, linear programming provides a technique to find where it is necessary to build extra capacity.

**Microchips**

The design of very large scale integration (VLSI) integrated circuits requires the laying of tracks on a printed circuit board. These tracks must not cross and must be as short as possible. Linear programming is used by VLSI design software to find the optimum layout of conductive tracks.

## Optimisation

**Activities**

- The teacher should discuss with the learner the solution of linear inequalities as illustrated in the students book
- The learner should do exercise 11

**Evaluation**: Give a written test on inequalities

**Linear optimization**

We can apply what we have learned above to linear optimization (also called linear programming), which is the process of finding a maximum or minimum value for some function under certain conditions (such as linear inequalities). Dealing with problems that involve linear optimization do not require you to learn any new skills; they simply require that you apply what you already know. So, let's move right to a practice problem.

***Example***

Find the maximum value of *y* given $-3x + 2y \leq 4$ and $x + y \leq 1$ subject to the condition that $x \geq 0$.

***Solution***

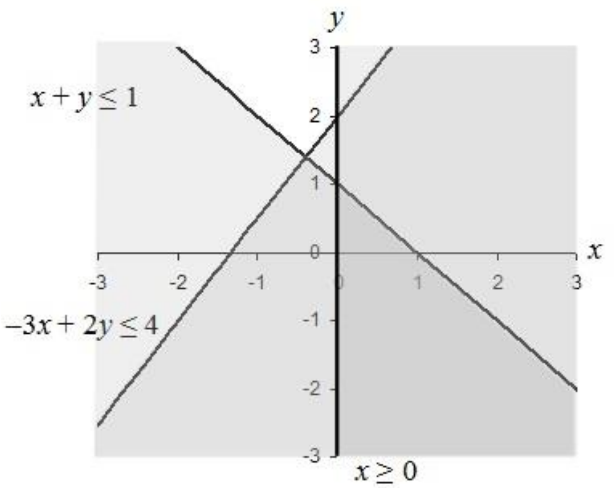
What we are given is a system of inequalities for which we must first find a corresponding solution set. Within this solution set, we can then find the maximum value of *y*. So, we can first apply what we already know: let's rearrange the inequalities into a form that we can easily graph.

$-3x + 2y \leq 4 \qquad x + y \leq 1 \qquad x \geq 0$

$2y \leq 3x + 4 \qquad y \leq 1 - x$

$y \leq 1.5x + 2$

Now, let's graph each of these inequalities, noting that we must use solid lines in each case.




The darkest shaded region (the wedge in the lower right of the graph) satisfies all the constraints on the problem. We then want to find the maximum value of *y*, which is clearly 1. (We can also find this value by substituting $x = 0$ into $x + y \leq 1$ and finding the maximum value of *y*, which likewise is clearly 1.)

## Exercise 2

***Solutions***

1. Draw the graph to provide the solutions:
$$(\mathbf{2},\mathbf{4}),(\mathbf{3},\mathbf{2}),(\mathbf{3},\mathbf{3}),(\mathbf{3},\mathbf{4}),(\mathbf{4},\mathbf{2}),(\mathbf{4},\mathbf{3}),(\mathbf{4},\mathbf{4}),(\mathbf{5},\mathbf{3}),(\mathbf{5},\mathbf{4}),(\mathbf{6},\mathbf{3}),(\mathbf{6},\mathbf{4}),(\mathbf{7},\mathbf{4}),(\mathbf{8},\mathbf{4})$$

2. a) $\mathbf{50x+5y\leq 120},\ \mathbf{400x+200y=10000},\ \mathbf{y>12},\ \mathbf{x\geq 8}$

   b)Draw the graph
   c)the objective function: $\mathbf{50x+70y}$ to be maximized
3. Inequalities: $\mathbf{y\geq 2x},\ \mathbf{x\leq 80},\ \mathbf{y\geq 20}$
   b) Graph
   c) Objective function: $\mathbf{50x+40y}$ to be maximized
4. a) Inequalities: $\mathbf{68x+44y\geq 520},\ \mathbf{x+y\geq 6},\ \mathbf{x>0},\ \mathbf{y>0}$

b) graph

c) The objective function $\mathbf{30\ 000}\boldsymbol{y} + \mathbf{20\ 000}\boldsymbol{y}$ to be minimized.

## Permutation and Combination formula

**Teaching Procedure**

At the end of the sub-unit, the students should be able to:

1. Define permutation, combination, and fundamental counting principle.
2. Work exercises on permutation.
3. Work exercises on combination
4. Differentiate between permutation and combination questions.

**Entry behaviour or previous knowledge:**

It is expected that the students have learnt the concept of factorial, so the teacher needs to test their entry knowledge by asking them some questions on the concept of factorial after which the teacher gives the definitions of the following:

**Permutation**

The number of ways that n elements can be arranged in order, is called a permutation of the elements.

Note: In permutation, every different ordering counts as a distinct permutation. For instance, the ordering $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e})$ is distinct from $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e})$, etc. that is, order of arrangement matters in permutation.

**Combination**

The number of ways objects could be combined. In combination, every different ordering does NOT count as a distinct combination. For instance, the committee {c, a, b} is the same as the committee {a, b, c} etc. that is order does not matter in combination.

**Fundamental counting principle**

If one event can occur in m ways and a second event can occur in n ways, then both events can occur in ways, provided the outcome of the first event does not influence the outcome of the second event. This can be stated in another way as:

If there are m roots of getting to a point $B$ from a point $A$, and there are n roots of getting to a point $C$ from $B$, then there are $m \times n$ roots of getting to the point $C$ from $A$ via $B$. For example, if there are 2 roots of getting to a point $B$ from a point $A$, and there are 3 roots of getting to a point $C$ from $B$, then there are $2 \times 3 = 6$ roots of getting to the point $C$ from $A$ via $B$.

**Exercise 3: Work in groups.**

1. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

A. 24 400
B. 21 300
C. 210
D. 25 200

2. In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there?

A. 159
B. 209
C. 201
D. 212

3. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there in the committee. In how many ways can it be done?

A. 624
B. 702
C. 756
D. 812

4. In how many different ways can the letters of the word 'OPTICAL' be arranged so that the vowels always come together?

A. 610
B. 720
C. 825
D. 920

5. In how many different ways can the letters of the word 'CORPORATION' be arranged so that the vowels always come together?

A. 47 200
B. 48 000
C. 42 000
D. 50 400

6. In how many ways can a group of 5 men and 2 women be made out of a total of 7 men and 3 women?

A. 1
B. 126
C. 63
D. 64

7. In how many different ways can the letters of the word 'MATHEMATICS' be arranged such that the vowels must always come together?

A. 9 800
B. 100 020
C. 120 960
D. 140 020

8. There are 8 men and 10 women and you need to form a committee of 5 men and 6 women. In how many ways can the committee be formed?

A. 10420
B. 11
C. 11760
D. None of these

9. How many 3-letter words with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?

A. 720
B. 420
C. None of these
D. 5040

10. In how many different ways can the letters of the word 'LEADING' be arranged such that the vowels should always come together?

A. None of these
B. 720
C. 420
D. 122

11. A coin is tossed 3 times. Find out the number of possible outcomes.

A. None of these
B. 8
C. 2
D. 1

12. In how many different ways can the letters of the word 'DETAIL' be arranged such that the vowels must occupy only the odd positions?

A. None of these B. 64

C. 120 D. 36

13. A bag contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the bag, if at least one black ball is to be included in the draw?

A. 64 B. 128

C. 32 D. None of these

14. In how many different ways can the letters of the word 'JUDGE' be arranged such that the vowels always come together?

A. None of these B. 48

C. 32 D. 64

15. In how many ways can the letters of the word 'LEADER' be arranged?

A. None of these B. 120

C. 360 D. 720

16. How many words can be formed by using all letters of the word 'BIHAR'?

A. 720 B. 24

C. 120 D. 60

17. How many arrangements can be made out of the letters of the word 'ENGINEERING' ?

A. 92 4000 B. 27 7200

C. None of these D. 182 000

18. How many 3 digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9 which are divisible by 5 and none of the digits is repeated?

A. 20
B. 16
C. 8
D. 24

19. How many words with or without meaning, can be formed by using all the letters of the word, 'DELHI' using each letter exactly once?

A. 720
B. 24
C. None of these
D. 120

20. What is the value of 100P2 ?

A. 9 801
B. 12 000
C. 5 600
D. 9 900

21. In how many different ways can the letters of the word 'RUMOUR' be arranged?

A. None of these
B. 128
C. 360
D. 180

22. There are 6 periods in each working day of a school. In how many ways can one organize 5 subjects such that each subject is allowed at least one period?

A. 3200
B. None of these
C. 1800
D. 3600

23. How many 6 digit telephone numbers can be formed if each number starts with 35 and no digit appears more than once?

A. 720
B. 360
C. 1420
D. 1680

24. An event manager has ten patterns of chairs and eight patterns of tables. In how many ways can he make a pair of table and chair?

A. 100
B. 80

C. 110 D. 64

25. 25 buses are running between two places P and Q. In how many ways can a person go from P to Q and return by a different bus?

A. None of these B. 600

C. 576 D. 625

26. A box contains 4 red, 3 white and 2 blue balls. Three balls are drawn at random. Find out the number of ways of selecting the balls of different colours?

A. 62 B. 48

C. 12 D. 24

27. A question paper has two parts P and Q, each containing 10 questions. If a student needs to choose 8 from part P and 4 from part Q, in how many ways can he do that?

A. None of these B. 6020

C. 1 00 D. 9 450

28. In how many different ways can 5 girls and 5 boys form a circle such that the boys and the girls alternate?

A. 2880 B. 1400

C. 1200 D. 3212

29. Find out the number of ways in which 6 rings of different types can be worn in 3 fingers?

A. 120 B. 720

C. 125 D. 729

30. In how many ways can 5 man draw water from 5 taps if no tap can be used more than once?

A. None of these B. 720

C. 60 D. 120

## Vectors

**Activities**

1. Teacher to guide the learners on how to get column vectors when given the coordinates of points
2. Teacher to discuss with the learner how to get the mid-point, magnitude and position vector.
3. Learners to be guided on how to identify parallel vectors
4. Learner to do exercise 12 and 13

**Evaluation**: Give a written test on vectors

**Exercise 4**

***Solutions***

1. a) $\begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$
   b) $\begin{pmatrix} \mathbf{2} \\ \mathbf{6} \end{pmatrix}$
   c) $\begin{pmatrix} \mathbf{-1} \\ \mathbf{-7} \end{pmatrix}$
2. a) 1.41
   b) $(\mathbf{2.5},\ \mathbf{0.5})$
3. a) $\begin{pmatrix} \mathbf{3} \\ \mathbf{3.5} \end{pmatrix}$
   b) $\begin{pmatrix} \mathbf{-2} \\ \mathbf{5} \end{pmatrix}$
   c) 5.39
4. a) $(\mathbf{3}, \mathbf{5.5})$
   b) 4.5

**Multiplication by a Scalar**

**Exercise 5**

***Solutions***

1. a) $\begin{pmatrix} 5 \\ 11 \end{pmatrix}$

   b)$\begin{pmatrix} -8 \\ 2 \end{pmatrix}$
   c)$\begin{pmatrix} 3 \\ 8 \end{pmatrix}$
   d)$\begin{pmatrix} -4.5 \\ 3 \end{pmatrix}$
   e) $\begin{pmatrix} -7 \\ 13 \end{pmatrix}$
   f) $\begin{pmatrix} 23.5 \\ 7 \end{pmatrix}$

2. a) $\begin{pmatrix} -8 \\ 3 \end{pmatrix}$

   b) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$
   c) $\begin{pmatrix} 2 \\ -3/2 \end{pmatrix}$
   d) $\begin{pmatrix} 3.5 \\ 2.5 \end{pmatrix}$
   e) $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$
   f) $\begin{pmatrix} 12 \\ 13 \end{pmatrix}$

3. a) False

   b)True
   c)False
   d)False
   e)True
   f)True

## Exercise 6

***Solutions***

1. a) 7.8

   b) 4.58
   c)10

2. i) 8.06

ii) $(\mathbf{0.5},\ \mathbf{4.1})$

3. i) $\begin{pmatrix} \mathbf{9} \\ \mathbf{5} \\ \mathbf{-1} \end{pmatrix}$

ii) $\begin{pmatrix} \mathbf{9} \\ \mathbf{9} \\ \mathbf{2} \end{pmatrix}$

iii) $\begin{pmatrix} \mathbf{-4} \\ \mathbf{-3} \\ \mathbf{-2.5} \end{pmatrix}$

iv) $\begin{pmatrix} \mathbf{16} \\ \mathbf{12} \\ \mathbf{0} \end{pmatrix}$

v) $\begin{pmatrix} \mathbf{-4} \\ \mathbf{12} \\ \mathbf{18} \end{pmatrix}$

4. i) $\begin{pmatrix} \mathbf{0.1} \\ \mathbf{-0.67} \\ \mathbf{0.33} \end{pmatrix}$

ii) $\begin{pmatrix} \mathbf{4.5} \\ \mathbf{-2.5} \\ \mathbf{4} \end{pmatrix}$

iii) $\begin{pmatrix} \mathbf{0.6} \\ \mathbf{-0.4} \\ \mathbf{0.6} \end{pmatrix}$

iv) $\begin{pmatrix} \mathbf{2.3} \\ \mathbf{0.33} \\ \mathbf{1.33} \end{pmatrix}$

v ) $\begin{pmatrix} \mathbf{18} \\ \mathbf{-3} \\ \mathbf{4.5} \end{pmatrix}$

vi) $\begin{pmatrix} \mathbf{-10} \\ \mathbf{-3} \\ \mathbf{-3} \end{pmatrix}$

5. a) i) $\boldsymbol{C}(\mathbf{1}, \mathbf{1.5}, \mathbf{4}),\ \boldsymbol{D}(\mathbf{5}, \mathbf{3.5}, \mathbf{1})$

ii) $\boldsymbol{CD} = \mathbf{5.39},\ \boldsymbol{AB} = \mathbf{10.77}$

b) i) $\boldsymbol{AB} = \mathbf{2}\boldsymbol{CD}$ **or** $\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{AB} = \boldsymbol{CD}$

ii)$\boldsymbol{OA = 2DE OR \frac{1}{2} OA = DE}$

6. $\boldsymbol{OQ} = \begin{pmatrix} \frac{19}{3} \\ \frac{16}{3} \\ \frac{-8}{3} \end{pmatrix}$

**UNIT 4**

# CALCULUS

<table>
<tr><td>Maths Secondary 4</td><td>Unit 4: Calculus</td></tr>
<tr><td>Learn about</td><td>Key inquiry questions</td></tr>
<tr><td>Learners should work in groups to solve problems using the derivatives of polynomial equations and the formulation of the equation of tangents and normals and points of maxima and minima.<br><br>Learners should find out how to calculate and apply maximum and minimum points and apply differentiation to kinematics.<br><br>They should works together in groups to solve problems involving integration, the application of integration and the integration of polynomials, and explain how the integration of polynomials is applied to finding the area under a curve using integration.</td><td>• How do we find the derivative of a polynomial and how can you formulate the equation of a tangent and normal to the curve at a point?<br>• How do we calculate and apply maximum and minimum values?<br>• How do we apply differentiation to kinematics?<br>• How is integration related to differentiation?<br>• How do we integrate polynomials and apply integration to find the area under a curve?</td></tr>
</table>

| Learning outcomes | | |
|---|---|---|
| Knowledge and understanding | Skills | Attitudes |
| • Understand the derivative of polynomial, equations of tangents and normals, maxima and minima points, application of differentiation to kinematics<br>• Understand integration, the application of integration, and the integration of polynomials to find the area under a curve | • Analyze, perform, investigate and calculate the maximum and minimum values<br>• Apply and solve equations of tangents and normals<br>• Apply integration of polynomials to find the area under a curve | • Be focused and resilient when solving problems of calculus |
| Contribution to the competencies:<br><br><u>Creative thinking and communication</u> in the application of differentiation and integration of polynomials.<br><br><u>Co-operation</u> and communication through teamwork | | |
| Links to other subjects:<br><br>Physics: finding velocity and acceleration<br><br>Business: estimating maximum and minimum profit<br><br>Geography: estimation of the area of a map | | |

**Objectives**

By the end of the topic, the learner should be able to:

- Find the derivative of a polynomial
- Find the equations of tangents and normal to a curve
- Sketch a curve
- Apply differentiation in kinematics
- Apply differentiation in finding maxima and minima of functions
- Integrate a polynomial.
- Apply integration to find the area of a curve

**Activities**

- Learners should work in groups with guidance from the teacher to determine the average rate of change between two points on a curve.
- The teacher should guide the learner in finding the gradient of a curve at a point, as in the students' book.
- Learner should do exercise 1.
- The teacher should guide the learners on how to find the equation of the tangent and normal to a curve.
- The teacher should lead the learner through examples provided in the book.
- Learner should do exercise 2.
- The teacher should discuss maxima and minima.
- The teacher should guide the learner on how to sketch curves.
- Learner should do exercise 2 the teacher should guide the student on the application of differentiation in calculating acceleration as shown in the students book.
- Learner should do exercise 3.
- The learner should be guided on how to establish the rule of integration
- The learner should do exercise 4.
- The teacher should discuss how to find the exact area under a curve
- Learner to do exercise 5.

**Evaluation**: oral and written tests on differentiation and integration should be given.

## Derivative of a polynomial

**Exercise 1**

***Solutions***

1. a) $\boldsymbol{y' = 12x}$

   b) $\boldsymbol{y' = 2x^5}$
   c) $\boldsymbol{y' = 2x^2}$
   d) $\boldsymbol{y' = -2^{11}}$
   e)$\boldsymbol{y' = 0}$
   f)$\boldsymbol{y' = 6}$
   g)$\boldsymbol{s' = 6t^2 + 8t - 6}$
   h)$\boldsymbol{s' = -30t^4 + 20t^3 - 3t^2}$

2. a) $\boldsymbol{18x^5 + \frac{3}{4}x^2 - 1}$

   b) $\boldsymbol{60x^2 - x^3}$
   c) $\boldsymbol{\frac{3}{4}x^2 - 6x + \frac{5}{2}}$
   d) $\boldsymbol{18x + 11}$

## Equation of a tangent and normal to the curve at a point.

**The teacher to review this topic from Secondary 3**

One fundamental interpretation of the derivative of a function is that it is the *slope of the tangent line* to the graph of the function.

The precise statement of this fundamental idea is as follows. Let $f$ be a function. For each *fixed* value $x_0$ of the input to $f$, the value $f'(x_0)$ of the derivative $f'$ of $f$ *evaluated at* $x_0$ is the slope of the tangent line to the graph of $f$ *at the particular point* $(x_0, f(x_0))$*on the graph*.

Recall the **point-slope form** of a line with slope mm through a point $(x_0, y_0)$:

$$\boldsymbol{y - y_0 = m(x - x_0)}$$

**In the present context, the slope is $f'(x_0))$ and the point is $(x_0, f(x_0))$, so *the equation of the tangent line to the graph of* ff *at* $(x_0, f(x_0))$ *is***

$$\boldsymbol{y - f(x_0) = f'(x_0)(x - x_0)}$$

The **normal line** to a curve at a particular point is the line through that point and *perpendicular* to the tangent. A person might remember from analytic geometry that the slope of any line *perpendicular to* a line with slope mm is the *negative reciprocal* −1/m−1/m. Thus, just changing this aspect of the equation for the tangent line, we can say generally that *the equation of the normal line to the graph of* $f$ *at* $(x_0, f(x_0))$ *is*

$$\boldsymbol{y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0).}$$

The main conceptual hazard is to mistakenly name the *fixed* point '$x$', as well as naming the *variable* coordinate on the tangent line 'xx'. This causes a person to write down some equation which, whatever it may be, is *not* the equation of a line at all.

Another popular mistake is to forget the subtraction $-f(x_0)$ on the left hand side. Don't do it.

So, as the simplest example: let's write the equation for the tangent line to the curve $y = x^2$ at the point where $x = 3$. The derivative of the function is $y' = 2x$, which has value $2 \cdot 3 = 6$ when $x = 3$. And the value of the function is $3 \cdot 3 = 9$ when $x = 3$. Thus, the *tangent line* at that point is

$$y - 9 = 6(x - 3)$$

The *normal line* at the point where $x = 3$ is

$$y - 9 = \frac{-1}{6}(x - 3)$$

So the question of finding the tangent and normal lines at various points of the graph of a function is just a combination of the two processes: computing the derivative at the point in question, and invoking the point-slope form of the equation for a straight line.

**Exercise 2**

*Solutions*

1. $y = -12x - 16$
2. Tangent $y = 5x - 10$, Normal $y = \frac{2}{5} - \frac{x}{5}$
3. $y = 4x - 1$
4. a) $y = 2$
   b) $8y = x - 49$

## Stationary points: Maximum and Minimum Values.

**Exercise 3**

***Solutions***

1. a) (2, 0) is minimum point
   b) $x = -4$ is a point of inflection, $x = 0$ is a minimum point
   c) Points of inflexion
   d) Maximum point
   e) Minimum point
   f) maximum point at $x = -3$, minimum point at $x = 2$

## Application to Kinematics: Calculation of velocity and acceleration.

Solutions to Exercise 4

1. a) $\mathbf{Velocity} = \mathbf{20}\boldsymbol{m/s}$ at $\boldsymbol{t} = \mathbf{1}$. $\mathbf{Velocity} = \mathbf{0}$at$\boldsymbol{t} = \mathbf{3}$

   b)$\mathbf{45}\boldsymbol{m}$

2. a) $\mathbf{6}\boldsymbol{t} - \mathbf{5}$

   b)$\mathbf{14}\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{m/s}$

3. $\mathbf{8}\boldsymbol{m/s^2}$

4. **a**) $-\mathbf{2}\frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{m/s}$

   b) 23m

5. a) $\mathbf{30}\boldsymbol{m/s^2}$

   b)$\mathbf{45}\boldsymbol{m/s}$
   c) $\mathbf{25}\boldsymbol{m/s^2}$

## Integration and Area under a curve

**Exercise 5**

***Solutions***

1. $\mathbf{21}\ \boldsymbol{units^2}$
2. $\mathbf{90.67}\boldsymbol{units^2}$
3. $\mathbf{19}\frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{units^2}$
4. a) For curve sketching the following must be clearly shown
   The x- intercepts at $\boldsymbol{x} = \mathbf{2}$ **and** $\boldsymbol{x} = -\mathbf{2}$
   The y-intercept at $\boldsymbol{y} = -\mathbf{4}$
   Turning point which in this case is the point $(\mathbf{0}, -\mathbf{4})$

b) The area$\mathbf{5}\frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{units^2}$