South Sudan

SECONDARY
4

Additional Mathematics

Teacher's Guide 4

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FOREWORD

I am delighted to present to you this Teacher's Guide, which is developed by the Ministry of General Education and Instruction based on the new South Sudan National Curriculum. The National Curriculum is a learner-centered curriculum that aims to meet the needs and aspirations of the new nation. In particular, it aims to develop (a) Good citizens; (b) successful lifelong learners; (c) creative, active and productive individuals; and (d) Environmentally responsible members of our society. This textbook, like many others, has been designed to contribute to achievement of these noble aims. It has been revised thoroughly by our Subject Panels, is deemed to be fit for the purpose and has been recommended to me for approval. Therefore, I hereby grant my approval. This Teacher's Guide shall be used to facilitate learning for learners in all schools of the Republic of South Sudan, except international schools, with effect from 4th February, 2019.

I am deeply grateful to the staff of the Ministry of General Education and Instruction, especially Mr Michael Lopuke Lotyam Longolio, the Undersecretary of the Ministry, the staff of the Curriculum Development Centre, under the supervision of Mr Omot Okony Olok, the Director General for Quality Assurance and Standards, the Subject Panelists, the Curriculum Foundation (UK), under the able leadership of Dr Brian Male, for providing professional guidance throughout the process of the development of National Curriculum, school textbooks and Teachers' Guides for the Republic of South Sudan since 2013. I wish to thank UNICEF South Sudan for managing the project funded by the Global Partnership in Education so well and funding the development of the National Curriculum, the new textbooks and Teachers' Guides. I am equally grateful for the support provided by Mr Tony Calderbank, the former Country Director of the British Council, South Sudan; Sir Richard Arden, Senior Education Advisor of DfID, South Sudan. I thank Longhorn and Mountain Top publishers in Kenya for working closely with the Ministry, the Subject Panels, UNICEF and the Curriculum Foundation UK to write the new textbooks. Finally, I thank the former Ministers of Education, Hon. Joseph Ukel Abango and Hon. Dr John Gai Nyuot Yoh, for supporting me, in my role as the Undersecretary, to lead the Technical Committee to develop and complete the consultations on the new National Curriculum Framework by 29 November 2013.

The Ministry of General Education and Instruction, Republic of South Sudan, is most grateful to all these key stakeholders for their overwhelming support to the design and development of this historic South Sudan National Curriculum. This historic reform in South Sudan's education system is intended to benefit the people of South Sudan, especially the children and youth and the future generations. It shall enhance the quality of education in the country to promote peace, justice, liberty and prosperity for all. I urge all Teachers to put this textbook to good use.

May God bless South Sudan. May He help our Teachers to inspire, educate and transform the lives of all the children and youth of South Sudan.

Deng Deng Hoc Yai, (Hon.)

Minister of General Education and Instruction, Republic of South Sudan

Table of Contents

Introduction 1

About this guide 2

LIMITS 4

Learning/Teaching Materials 5

Learning/Teaching 5

Assessment 8

Estimating the limit graphically 10

Determining limits by direct substitution 13

Determining limits by indirect substitution 14

Continuity of a function 15

Continuity on an interval 17

Limits of trigonometric functions 19

Limits using the squeeze theorem 21

Calculating trigonometric limits 22

Limits at infinity 23

Limit and gradient of curves 23

Limits and series 24

Limits and the area under a curve 25

TRIGONOMETRY 2 27

Learning/Teaching Materials 28

Learning/Teaching Activities 29

Assignment 29

Solution of simple trigonometric equations 30

Solving quadratic trigonometric equations 34

Graphical solutions of Quadratic Trigonometric Equations 34
Using trigonometric identities in solving equations with more than one trigonometric function 35

CALCULUS 3 37
Learning/ Teaching Materials 39
Learning/Teaching Activities 39
Assessment 41
The product rule 41
The quotient rule 43
Differentiation by the product and quotient rule 44
Derivative of exponential and logarithmic functions 46
Derivatives of trigonometric functions 47
Integration 48
Integration of exponential and logarithmic functions 52
Integration of trigonometric functions 52
Application of integration to kinematics 55

PARTIAL FRACTIONS 56
Learning/Teaching Activities 57
Partial Fractions of linear factor(s) in Denominator 58
Partial Fractions of Quadratic Factor(S) In the Denominator. 58
Application of Partial Fractions 59
Expressing Partial Fractions as a single Fraction 59
Assessment 59
Partial Fractions with linear factor(s) in the denominator 61
Partial Fractions of functions with a quadratic expression in the denominator 61

Application of partial fractions in integration 62

VECTORS 63
Learning/Teaching Materials 63
Learning/Teaching Activities 65
Positional Vector in 3-Dimensions 65
Addition of vectors in Cartesian plane 65
Magnitude of Vector 66
Scalar Multiplication of Vector 66
Vector Equation of a line 66
Cartesian equation of a straight line. 67
The Scalar Product (The dot product) 67
Scalar products of two vectors on a Cartesian plane 67
Application of scalar product 68
Assessment 68
Position vectors in 3 dimensions 69
Addition of vectors 69
In 3 dimensions 71
Magnitude of vector (Length of a line and vector) 72
The vector equation of line 75
Cartesian form of the equation of straight line 77
The scalar product 78
Scalar product of two vectors on Cartesian plane 79
Application of scalar products 81
Finding the angle between two vectors 81

COMPLEX NUMBERS 84
Learning/teaching Materials 84
Learning/ Teaching Activities 86
Graphical Representation of complex number 86
Polar form of a complex number 86
De Molvre's Theorem 87
Solving quadratic Equations with complex solutions 87
Assessment 87
Polar form of a Complex number 87
De Moivre's Theorem and the roots of complex numbers 92

Introduction

Teaching Mathematics is taking place in rapidly changing conditions. It is necessary to look for optimal didactic and educational solutions encompassing goals and contents as well as forms and teaching methods allowing for preparing students to face the challenges of the contemporary world.

The most significant role of educational system in terms of teaching Mathematics is developing and promoting subject competences as an important factor fostering student's personal development and the development of society. Well organised mathematical education facilitates logical thinking and expressing ideas, organizing own work, planning and organizing the learning process, collaboration and responsibility; it prepares for life in a modern world and enables to perform many jobs.

The teacher is required to pay more attention to students' awareness of developing learning skills and study habits, recognizing and analysing problems and predicting solutions to them. Undeniably, the implementation of modern teaching methods and techniques enhances students' curiosity about Mathematics and increases their understanding of the basis of mathematical and scientific knowledge. In accordance with the trends teaching Mathematics is supposed to help students understand and solve everyday problems.

The aim of teaching Secondary Mathematics is to encourage contemporary students to work in class, acquire knowledge and skills that are necessary in life. Moreover, research shows that teachers applying active methods assess the effectiveness of their work and how students respond to this way of teaching.

About this guide

The purpose of this guide is to offer suggestions that are helpful to Secondary 4 Mathematics teachers on planning, organizing, executing and evaluating the learning and teaching of mathematics. The suggestions will serve as useful starting points to the teachers who are expected to be dynamic innovative and creative to make the leaning process fit the learners.

The guide is to be used alongside Mathematics Students book for secondary 4. It consists of 6 units, in line with Additional Secondary 4 mathematics syllabus.

Each of the unit is structured to contain:

1. Introduction
2. Objectives
3. Teaching/Learning Activities
4. Answers to the exercises given in student's secondary 4 book.

In each case, the introduction highlight the relevant work than learners are expected to have covered in their previous mathematics units and what they are expected to have covered previously. It also highlights what they are expected to cover in the unit. The teacher is expected to make a quick lick up of previously learnt concepts. Learners should be able to make relevant references to their previous work. Where possible the mathematics teach ma make an entry behavior evaluation as a revision on previously learnt units related to the unit under study.

The unit objectives specify the skills (cognitive, affective, and psychomotor) that teachers will use to enable learners understand each unit. The objectives are likely to serve a useful purpose if they when stated to reflect the local conditions of the learner. For example, the type of students and the available learning resources. The teacher may break down the unit objectives to various objectives that enhance the learners understanding of the process involved and to suit different situations in the lesson, schools, society and the world at large.

Teaching/learning activities highlight the most noticeable and important. Points encountered in the learning process and suitable techniques to be used in handling each objective(s).

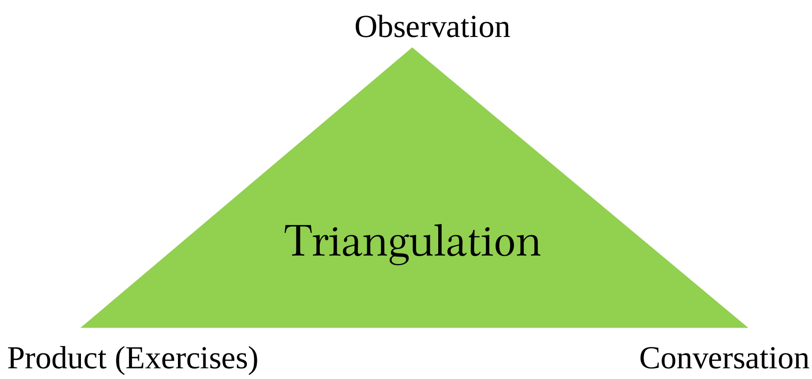
Answers to each exercise in the students' book are provided in these teachers guide. It is contemplated that the most conducive and favorable outcome from the guide will be realized if other sources of learning mathematics are properly organized and used.

Among others, the following should be used alongside the guide:

1. The Schemes of work
2. The teacher's Lessons plans.
3. The Records of the work covered by the learners.

Making Classroom Assessment

- Observation – watching learners as they work to assess the skills learners are developing.
- Conversation – asking questions and talking to learners is good for assessing knowledge and understanding of the learner.
- Product – appraising the learner's work (writing report or finding, mathematics calculation, presentation, drawing diagram, etc.).



To find these opportunities, look at the "Learn About' sections of the syllabus units. These describe the learning that is expected and in doing so they set out a range of opportunities for the three forms of opportunity.

UNIT 1

LIMITS

Additional Math Secondary 4	**Unit 1:Limits (rational functions)**
Learn about	**Key inquiry questions**
Learners should investigate the concept of limits, limits of polynomial functions, theorems of limits through discussion. They should find limits of rational functions and techniques of evaluating limits of rational functions when the result of direct substitution is undefined; such as $\mathbf{0}/\mathbf{0}$ and ∞/∞. Learners should investigate important limits such as $\lim\limits_{x\to a}\frac{x^n-a^n}{x^m-a^m}$ and $\lim\limits_{x\to 0}\frac{\sin x}{x}$ to be able to solve problems involving limits individually and in groups. They should supplement their learning by using the internet, reference textbooks and working with others.	• How do we determine limits of polynomials and rational function? • How can we evaluate limits of a function when the result is undefined? • How can we investigate the use of some important limits to derive more formulae? • In which ways can limits be used in solving problems?

Learning outcomes

Knowledge and understanding	**Skills**	**Attitudes**

• Limits of rational functions	• Work out problems involving limits • Use theorems of limits to derive formulae • Use the theorem of limits to solve real life problems • Investigate methods of finding limits	• Appreciate the concept and use of limits in calculus • Develop curiosity when solving problems of limits • Build an understanding of the theorems of limits • Team work
Contribution to the competencies: Critical thinking: Investigate and analyse concepts of limits Communication: Communicate the concepts and theorems of limits coherently and clearly to others Cooperation: Discuss theorems of limits in groups		
Links to other subjects: Physics: Mechanics – instantaneous velocity and acceleration Economics: rate of change of production. ICT: Internet		

Learning/Teaching Materials

Scientific calculator, a wire/thread, nails, graph paper, graph board, geometrical set, a computer, rope, water tank and spring etc.

Learning/Teaching

1. Ask learners to perform an activity using a strength 24m long in a field and four nails till they form a square of the maximum area. Ask learners

guiding question so as to complete a table similar to the one in student's book. Lead them in a discussion on the existence of a limit.

2. Guide learners on drawing graphs of limits.
3. Guide learners to come up with their own definition of limits.
4. Guide learners in discussing how to eliminate limits;
    a) Numerically
    b) Graphically
    c) By substitution
5. Demonstrate the use of a scientific calculator in completing the table of limits using a scientific calculator.
6. Make visual illustrations and discuss how a continuous point of a curve is illustrated.
7. Drawing graphs of functions which are not continuous.
8. Drawing graphs of function defined by two expressions as illustrated in students books and figure 1.5.
9. Discussion of limits and illustrations of graphs of limits that exist when a function is not defined.
10. Making a case study of real life situations that lead to different forms of limit.
11. Making a field study and guided discussion and problem solving in real life areas where limit exist. Such areas may include in business sales, marketing, salaried relationships and health analysis.
12. Discussion on an oscillating activity such as nature of swinging a tied tread so as to form wave, developing wave on still water in a tank or beaker and by probing learners develop concept of oscillating curve limits.

13. Discussion on determining limits by direct substitution of constants into an expression.
14. Guide learners on determining limits in situation that do not require direct substitutions. emphasis should be made on undetermined situation of a fraction such as;
    i. $\frac{0}{0}$, *zero numerator and denominator*
    ii. $\frac{x}{0}$, *variable numerator and zero denominator*
    iii. $\frac{\infty}{\infty}$, *infinity numerator and denominator*
15. Discussion of determining by rationalizing part of the expression so as to remove the undefined forms of the expressions.
16. Discussion on limit of trigonometric expressions and their simplification using ;
    i. $\frac{\sin x}{x}$
    ii. $x^2 \sin\left(\frac{1}{x}\right)$
17. Guide learners through visual illustration and analysis of one side limit using the example on student's book.
18. A case study on shipment and real life existence of limit of one side is illustrated using real life scenarios.
19. Drawing of graphs that illustrate continuity of a function is made.
20. Teacher guide students by use of visual illustrations in describing continuity of an interval at an open and at a closed interval.
21. Defining and determining limit at infinity by drawing of graphs, completing numerical tables, by substitution or dividing throughout as illustrated in the student's book.

22. Discussion on application of limits in determining gradient of curves. Making visual illustrations relating limits and gradient of curves.
23. Making a case study on application of limits and gradients and applying it in problem solving.
24. Guiding learners on use of limits to determine the area under a curve. Making visual illustration on area under a curve. Defining area as an expression of limit and discussing its application in solving real life problems.

Assessment

The learner should be able to define terminologies defined in students' book. Ensure they are able to solve questions in the exercise and can solve real life problems by applying limits.

ANSWERS

Exercise 1.1

Work in groups of four students.

Determine the limit of the following at the described positions numerically.

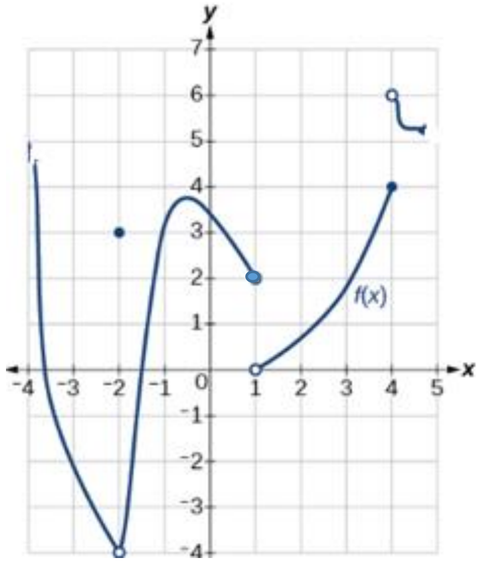
1. $\lim\limits_{x \to -1} \quad (x^2 + x - 8)$
2. $\lim\limits_{x \to -1} \frac{(x^2+x-6)}{x+3}$
3. $\lim\limits_{x \to -1} \frac{(x+1)}{x^2-x-2}$
4. $\lim\limits_{x \to 1} \quad (5x + 8)$
5. $\lim\limits_{x \to 0} \quad 2x^2 + 3x + 3$
6. $\lim\limits_{x \to 2} \frac{x^2-4}{x-2}$
7. $\lim\limits_{x \to 1} \quad 6x$
8. $\lim\limits_{x \to 4} \sqrt{x}$
9. $\lim\limits_{x \to 3} \quad (x + 4)^2$

Solutions

1. 2
2. -3
3. 0
4. 13
5. 3
6. 4
7. 7.6
8. 2
9. 49

Estimating the limit graphically

Task 1



Use the graph to determine the following limits, if they exist, giving reasons for your answer.

i. $\lim_{x \to -2} f(x)$

ii. $\lim_{x \to 1} f(x)$

iii. $\lim_{x \to 4} f(x)$

Solution

a) $\lim_{x \to -2} f(x) = -4$ the function approach -4 from both sides

b) $\lim_{x \to 1} f(x)$

Does not exist since the value of $f(x)$ approaches two values, on L.H.S approaches 2 and that from R.H.S approaches 1. They should approach the same figure.

c) $\lim_{x \to 4} f(x)$
Does not exist, the function approaches two values, 4 and 6, from both sides of the functions.

Task 2

a. In pairs, draw the graph of,

$$f\,(x) = \begin{cases} x+1 & for \quad x < 1 \\ \frac{1}{2}(9-x) & for \quad x > 1 \\ 3 & \text{for } x = 1 \end{cases}$$

b. ***Using the graph find***
i. $\lim_{x \to 0} f(x)$
ii. $\lim_{x \to 1} f(x)$

Solution

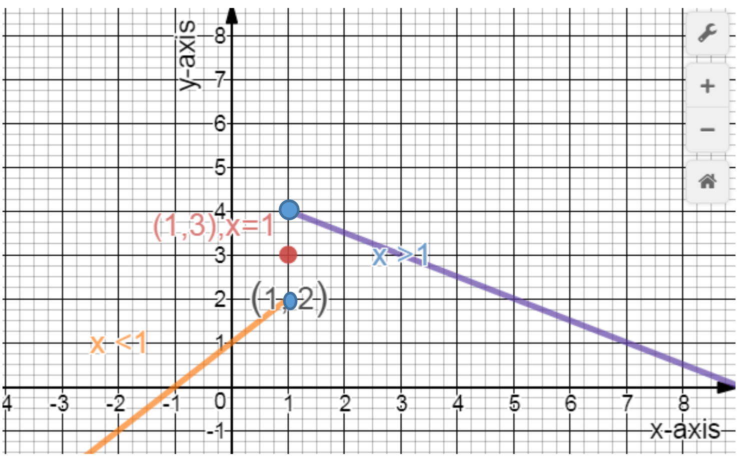


Figure 1.6

i. $\lim_{x \to 0} f(x) = 1$

ii. $\lim_{x \to 1} f(x)$ does not exist since the value of 3, is not approached from values of $x < 1$ or $x > 1$

Exercise 1.2

Learners to complete the exercise in groups.

Solutions

1.
   a) 2
   b) 2
   c) Does not exist
   d) Does not exist
   e) 2

2.
   a. 4
   b. 5
   c. 0

3
   a) 1
   b) Does not exist
   c) 4
   d) Does not exist

4. 12
5. 4
6. Does not exist as x c $f(x)$ has 2 values.
7. 1
8. Does not exist it is unbounded.
9. Does not exist. It is unbounded.

Determining limits by direct substitution

Task

In groups, discuss and complete the table below.

General result	**Explanation**
$\lim\limits_{x\to c} a = a$	The limit of a constant a is the same constant (a). This is because the value of x does not affect the value of the function $f(x) = a$.
$\lim\limits_{x\to c} x = c$	
$\lim\limits_{x\to c} af(x) = a\lim\limits_{x\to c} f(x)$	
$\lim\limits_{x\to c}(f(x) + g(x)) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$	
$\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$	
$\lim\limits_{x\to c} f(x).g(x) = \lim\limits_{x\to c} f(x).\lim\limits_{x\to c} g(x)$	
$\lim\limits_{x\to c} x^n = \left(\lim\limits_{x\to c} x\right)^n$	

Determining limits by indirect substitution

Exercise 1.3

In pairs, find the solution to the following limits using any suitable method

1. $\lim_{x\to 3} (x+4)^2$
2. $\lim_{x\to 1} x^2 + 2x + 3$
3. $\lim_{x\to 2} 2x^2 + 4$
4. $\lim_{x\to 1} \frac{x^2+3x+4}{x}$
5. $\lim_{x\to 5} (10-x^2)$
6. $\lim_{x\to 1} \frac{1}{2} x^3+x^2+5$
7. $\lim_{x\to 3} \frac{5x+3}{x^2 1}$
8. $\lim_{x\to \pi} 3sinx$
9. $\lim_{x\to \pi} (\tan x + \cos x)$
10. $\lim_{x\to 3} \sqrt{x^2} + n$
11. $\lim_{x\to 2} \frac{x-2}{x^2+2x+1}$
12. $\lim_{x\to 4} x^x$
13. $\lim_{x\to 2} 4^x$
14. $\lim_{x\to 1} 7^{(x+1)}$
15. $\lim_{x\to 2} \sqrt[3]{x^2+2}$
16. $\lim_{x\to 0} \frac{2x^2+4}{x+1}$
17. $\lim_{x\to 1} \frac{x^4+16}{x^2+1}$
18. $\lim_{x\to 3} e^{2x}$
19. If f(x) = $\frac{2x^2+x}{x^2-4}$, find $\lim_{x\to 1}$ f(x)

Solutions

1. 6
2. 12
3. 8
4. 15
5. 6.5
6. 1.8
7. 0
8. 1
9. 4
10. 0

11. 256
12. 16
13. 49
14. 3
15. 4
16. 7.5
17. 403.42
18. -2/3

Exercise 1.3

Students to work in groups.

Solutions

1. -5
2. 4
3. 0
4. ½
5. 8
6. 1/3
7. 0
8. 0
9. 11/2
10. 0

Exercise 1.4

Students to work individually.

Solutions

1. 5
2. 2
3. ¼
4. 0

Continuity of a function

Task

Determine whether the functions $f(x), g(x)$ and $h(x)$ are continuous at $x = 1$.

a) $f(x) = \frac{x^2-1}{x-3}$

b) $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

c) $h(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

Solution

a) $f(x) = \frac{x^2-1}{x-1}$

Is $f(x)$ defined? Substitute $x = 1$

$f(1) = \frac{0}{0}$ which is not defined hence not continuous.

We do not need to verify the 2nd and 3rd condition since the first has failed.

b) $f(x) = \frac{x^2-1}{x-1}$

1st is $g(x)$ defined at $g(1)$? Yes, from 2nd expression $g(1) = 2$.

2nd does $\lim_{x\to 1} g(x)$ exist? $g(x) = \frac{x^2-1}{x-1} = x+1$, provided $x \neq 1$

$\lim_{x\to 1} g(x) = 1 + 1 = 2$, yes it exist.

3rd – is $g(a) = \lim_{x\to a} g(x)$

$g(1) = 2$

$\lim_{x\to 1} g(x) = 2$

Yes they are the same. The three conditions are satisfied hence $g(x)$ is continuous when $x = 1$.

c) $h(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 3 & x = 1 \end{cases}$

1st is $h(x)$, $h(1)$ defined? Yes

h=3

2nd does $\lim_{x \to 1} \frac{x^2-1}{x-1}$ exist? Yes, $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} x + 1 = 1 + 1 = 2$

3rd if $h(x) = \lim_{x \to a} h(x)$

$h(1) = 3$

$\lim_{x \to 1} h(x) = 2$

The 3rd condition is not met hence $h(x)$ is not continuous when $x = 1$.

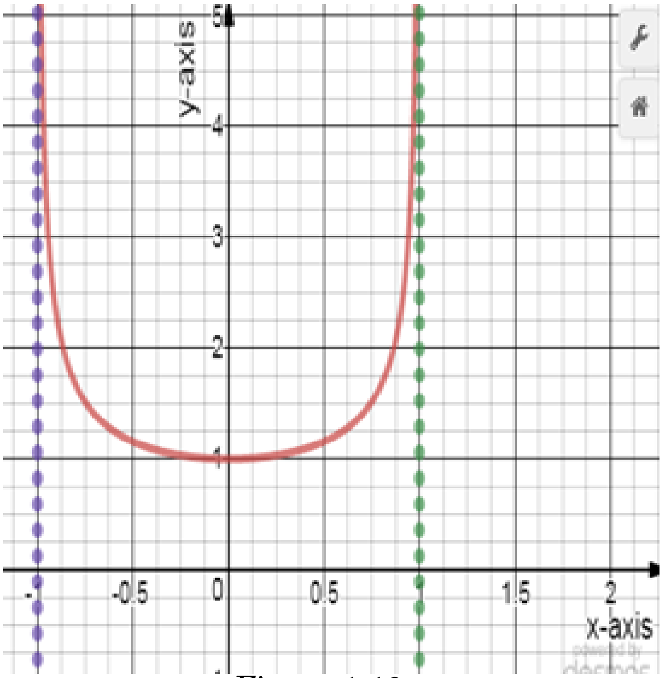
Continuity on an interval

Task

The graph below shows $= f(x) = \frac{1}{\sqrt{1-x^2}}$. Determine whether the function is continuous on the

a) open interval (-1, 1)

b) closed interval $[-1, 1]$



$$f(x) = \frac{1}{\sqrt{1-x^2}}$$

Figure 1.13

Solution

Clearly all values of open interval (-1,1) are defined.

$\lim_{x\to 1^-} f(x)$ is undefined.

$\lim_{x\to -1^+} f(x)$ is undefined.

Hence $f(x) = \frac{1}{\sqrt{1-x^2}}$ is continuous at open interval (-1, 1) only. At closed interval $[-1,\ 1]$ it is not continuous.

Exercise 1.5

Solutions

1. Not Continuous

2.
   a. Continuous
   b. Continuous
3. not continuous
4.
   a. Not continuous
   b. Continuous.
5. Closed. Open
6. 1
7.
a. None
b. 2
c. 5

Limits of trigonometric functions

Task 1

In pairs

Find the number of degrees equivalent to 1 radian correct to 3d.p.

Copy and complete the following table

θ°	0		45		90		120											330		
θ^{c}		$\frac{\pi}{6}$			$\frac{\pi}{2}$						π	$\frac{7\pi}{6}$								2π

Task 2

i. Draw an isosceles right angled triangle and an equilateral triangle.
ii. Label all the angles in the triangle.
iii. Copy and complete the following table

Angle in degrees(θ°)			45		90
Angle in radians (θ^{c})		$\frac{\pi}{6}$			$\frac{\pi}{2}$

$\sin\theta$				$\frac{\sqrt{3}}{2}$	
$\cos\theta$	1				
$\tan\theta$					

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	$0°$	$30°$	$45°$	$60°$	$90°$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Task 3

Sketch the graphs of the trigonometric functions using a horizontal axis with $-2\pi \leq \theta \leq 2\pi$.

State the limits for these functions: $\lim_{\theta \to a} f(\theta)$

Task 4

Use the graph below to determine $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$

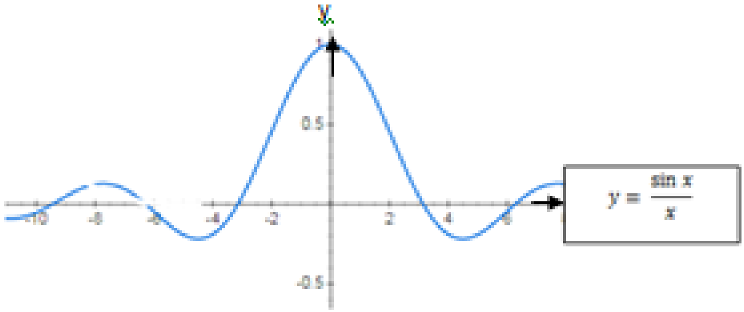


Figure 1.9

Limits using the squeeze theorem

Task

In groups, study figure 1.19 below that shows three functions and answer the questions that follow.

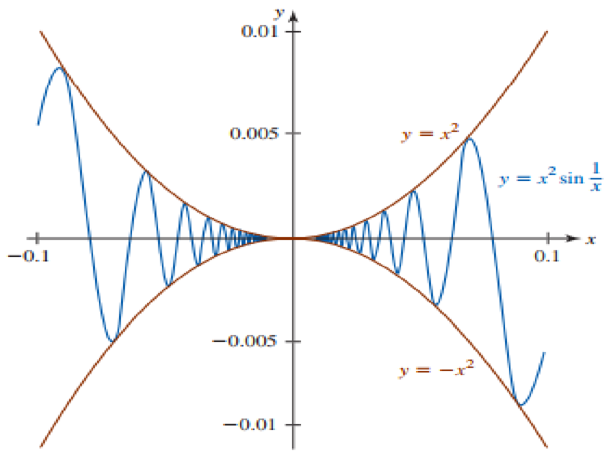


Figure 1.19

a. Determine ,

i. $\lim_{x \to 0} x^2$

ii. $\lim_{x \to 0} -x^2$

b. Of the three functions, which function is squeezed by the two other functions?

c. Using the graph infer $\lim_{x \to 0} x^2 \sin\frac{1}{x}$

Solution

i. $\lim_{x \to 0} x^2 = 0$

ii. $\lim_{x \to 0} -x^2 = 0$

$$\lim_{x \to 0} x^2 = \lim_{x \to 0} -x^2 = 0$$

In the figure $g(x) = x^2$ and $h(x) = -x^2$, they have the same limit when $x = 0$ and squeeze $f(x) = x^2 \sin\frac{1}{x}$ between them. Using the squeeze theorem $x^2 \sin\frac{1}{x}$ has the same limit when $x = 0$.

Therefore, $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$.

Calculating trigonometric limits

Exercise 1.6

Solutions

1. 1
2. ¼
3. 0
4. -1/3
5. 0
6. 0
7. 1
8. 1
9. 1
10. 1
11. 3

Limits at infinity

Exercise 1.7

Solutions

a) 40
b) 0
c) 3
d) Does not exist

2.

A. 2
B. 2T= 0.5X +5000
C. (0.5X +500)/X
D. 0.5

Limit and gradient of curves

Exercise 1.8

Solutions

1. 4
2. a) $y' = 2x$
   b) -2
   c) 4

3. a)
b) $\frac{d(\sin x)}{dx} = \cos x$
c) $\frac{d(\cos x)}{dx} = -\sin x$

Limits and series

Task

In groups, establish the following results:

1. $\sum_{i=1}^{n} c = cn$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Use these results to find

1. $\sum_{i=1}^{100} i$

Solution

$$\sum_{i=1}^{100} i = \frac{n(n+1)}{2}$$

$$= \frac{100(100+1)}{2}$$

$$= 50 \times 101$$

$$= \underline{\underline{5050}}$$

1. $S = \sum_{i=1}^{n} \frac{i+2}{n^2}$

Solution

a) $S = \sum_{i}^{n} \frac{i+2}{n^2}$

$$S = \frac{1}{n^2} \sum_{i}^{n} (i+2)$$

$$S = \frac{1}{n^2}\left(\sum_{i}^{n} i + 2n\right), \qquad but \sum_{i}^{n} i = \frac{n(n+1)}{2}$$

$$S = \frac{1}{n^2}\left(\frac{n(n+1)}{2} + 2n\right)$$

$$S = \frac{1}{n^2}\left(\frac{n^2 + n + 4n}{2}\right)$$

$$S = \frac{1}{n^2}\left(\frac{n^2+5n}{2}\right) = \frac{n(n+5)}{2n^2}$$

$$\therefore s = \frac{n+5}{2n}$$

Limits and the area under a curve

Exercise 1.9

Solutions

1.

a.

$$\frac{n^3}{3} + \frac{n^2}{2} + \frac{25n}{6}$$

b.

c. $\frac{8n^3}{3} + 4n^2 + \frac{4n}{3}$

a. $9n^2$+9n

d. n^2+6n

2.

a. 3

b. $\frac{21}{4}$= 5.25

3. 18 $\frac{2}{3}$ cm

4. $\frac{22}{7}$

UNIT 2

TRIGONOMETRY 2

Additional Math Secondary 4		**Unit 2:Trigonometry**
Learn about		**Key inquiry questions**
Learners should revisit their knowledge in trigonometry to simplify trigonometric ratios (identities), solve trigonometric equations through discussion and investigate their uses in solving problems. They should investigate the factor formulae (sum and difference of two angles), trigonometric function acos θ+ bsin θ and equation acos θ+ bsin θ=c through demonstration by the teacher using worked examples, and explore the idea of equations of trigonometry to solve problems individually and in groups. They should supplement their learning by using the internet, reference textbooks and working with others.		• Why is it important to learn trigonometry? • How can we simplify and solve trigonometric equations? • When do we use the form acos θ+ bsinθ to solve trigonometric equations? • How can we derive identities from the factor formulae? • How can we use the equations of trigonometry to solve real life problems?
Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**

• Simplification of trigonometrical ratios and solutions of trigonometrical equations • Sum and differences of two angles (A±B) • Functions, $a\cos\vartheta + b\sin\vartheta$ • The equation, $a\cos\vartheta+b\sin\vartheta =c$	• Investigate factor formulae. • Use equation $a\cos\theta+ b\sin\theta=c$ to solve trigonometric problems. • Apply trigonometric equations to solve real life problems. • Investigate the use of trigonometric functions in real life situation.	• Show interest and determination to techniques of proving identities. • Appreciate the use of trigonometry in problem solving. • Show resilience and responsibility in working with trigonometry. • Teamwork
Contribution to the competencies: <u>Critical thinking</u>: Analyse and carry out investigation in problem solving related to trigonometry <u>Cooperation</u>: Collaborate with others in solving difficult problems		
Links to other subjects: Physics: Trigonometric identities used in waves TVET: Technical drawings/Engineering/Woodwork Geography: Navigation ICT: Internet		

Learning/Teaching Materials

Scientific calculator, graph book, graph board.

Learning/Teaching Activities

1. Guide learners in a discussion to review main concepts discussed in previous unit of trigonometry as illustrated in the student's book.
2. Solving trigonometric equation by discussion and guided questioning technique by use of algebraic calculations.
3. Drawing graphs of trigonometric curves after discussion and use of think-pair-share approach.
4. Discussion on solving trigonometric equations by either numerical or graphical method.
5. Learners are guided on solving quadratic trigonometric equation using identities in mathematical table.
6. Teacher emphasizes on the use of different units in solving trigonometric equations.
7. Solving quadratic equations with more than one trigonometric functions by guided discussion using mathematical table as shown in student's book.
8. Discussion on the suitable method of solving quadratic trigonometric equations. Teacher to oversee and supervise the discussion and to give remarks after the discussion.
9. Guide learner into a case study and a field work on application of trigonometry and in real life scenarios.

Assignment

Students should be able to solve all the question in learners book exercise. Any improvement needed should be done immediately.

ANSWERS

Solution of simple trigonometric equations

Task

a. In pairs, draw the graph of the following functions on the same set of axes,
   i. $y = \cos x$
   ii. $y = 0.5$

b. Use the graphs to solve $\cos x = 0.5$ for $-180^0 \leq x \leq 360^0$

Solution

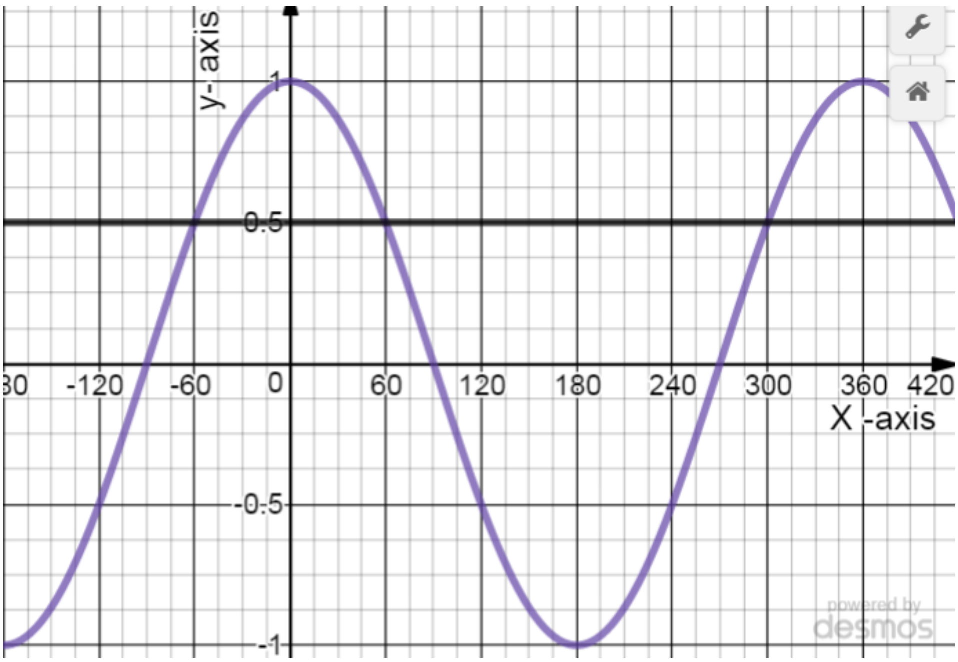


Figure 2.3

Choosing the intersections of the graphs as the solutions of x we get

$x = -60^0, 120^0$ and 240^0

Task

In groups of three, sketch the graph of $y = sin\,\mu$ where $\mu = 2x$ and solve $sin\,2x = \frac{1}{2}$, $0 \leq x \leq 2\pi$

Solution

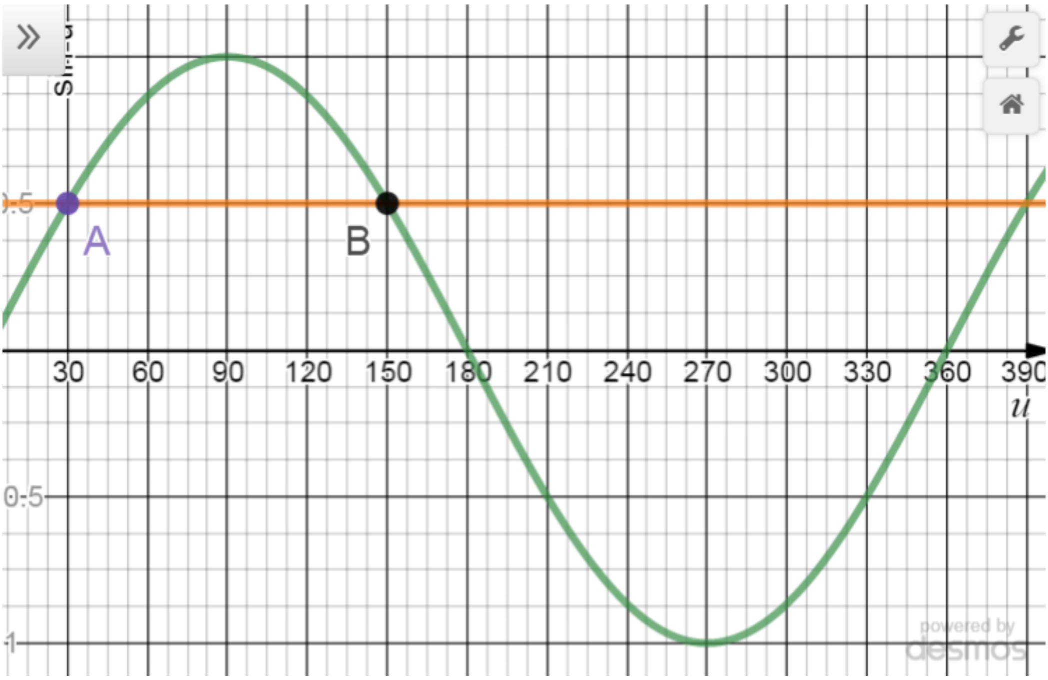


Figure 2.5

From graph it is clear that;

$\mu = \frac{\pi}{6}, \frac{5\pi}{6}$ *and* $\frac{13\pi}{6}$.

Since $\mu = 2x$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}$ *and* $\frac{13\pi}{6}$ Dividing by 2.

$x = \frac{\pi}{12}, \frac{5\pi}{12}$ and $\frac{13\pi}{12}$.

Exercise 2.1

Work in groups.

1. Find the solution to the trigonometric equations below by using graphs of respective equations.
   a) $Tan\ x = -1 \quad -\pi \leq x \leq \pi$
   b) $Cos2x = \frac{\sqrt{3}}{2} \qquad 0 \leq x \leq 2\pi$
   c) $Cos\ 2x = \frac{1}{\sqrt{2}} \qquad -180^0 \leq x \leq 180^0$
   d) $Tan\ 2x = 1 \qquad -90^0 \leq x \leq 90^0$
   e) $Sin\ 2x = \frac{1}{2} \qquad -180^0 \leq x \leq 0$
   f) $Cos\left(\frac{1}{2x}\right) = -5\frac{3}{2} \qquad -180^0 < x < 180^0$

2. Find the solution of the following trigonometric equations using Desmos graphs.
   a) $Sin\ (x)\ + 2 = 3$ for $0^0 < x < 360^0$
   b) $7\ tan\ \theta = 2/3 + tan\theta$ for $0° < x < 360^0$
   c) $3(\sin x + 2) = 3 - \sin x \qquad 0° < x < 360^0.$
   d) $\frac{1}{2}(\sec\theta + 3) = \sec\theta + \frac{5}{2}. \qquad 0° < x < 360^0$
   e) $Sin\ (\frac{x}{2}) = \frac{\sqrt{3}}{2} \qquad \frac{-\pi^c}{2} \leq x \leq \pi^c$
   f) $\tan x = \sqrt{3}. \qquad -180^0 \leq x \leq 180^0$
   g) $\tan x = -\sqrt{3} \qquad -180^0 \leq x \leq 180^0$
   h) $Cos\ x = \frac{1}{2} \qquad -180^0 \leq x \leq 180^0$
   i) $Sin\ x = -\frac{1}{2}. \qquad -180^0 \leq x \leq 180^0$
3. Using any suitable method solve the following trigonometric equations.
   a) $\tan^3\theta + 3 = 0 \qquad 0° \leq x \leq 360^0$
   b) $\tan x = \sqrt{3 \qquad\qquad} \ 30^0 < x < 2\pi^c$
   c) $\sin 2x = -1 \qquad -\pi^c < x < \pi^c$

d) $\cos 3x = \frac{1}{\sqrt{2}}$ $\quad - \pi^c \leq x \leq \pi^c$

e) $\tan \frac{1}{2} x = -1$ $\quad - 2\pi^c < x < 0$

f) $\sin x = \frac{1}{\sqrt{2}}$ $\quad 0° \leq x \leq 360^0$

g) $\cos x = \frac{-1}{2}$ $\quad 0° \leq x \leq 360^0$

h) $\tan x = \frac{1}{\sqrt{3}}$ $\quad 0° \leq x \leq 360^0$

i) $\cos x = -1$ $\quad 0° \leq x \leq 360^0$

4. The voltage v in volts in an electrical circuit given by the formula $v = 20 \cos(\pi^c)$ where t is time in seconds.
   a) Draw a graph of voltage v against time for $0 \leq t \leq 2$ intervals of $\frac{\pi}{12}$
   b) What is the voltage of the electric circuit when 1 sec?
   c) How many ties does the voltage v equal 12 voltage in the first two seconds?
   d) At what time is the voltage 12v?

Solutions

1. a) $-\frac{\pi}{4}, \frac{3\pi}{4}$

b) $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

c) -157.5^0, -22.5^0, 22.5^0, 157.5^0

d) -45^0, 15^0, 75^0

e) -165^0, 105^0

f) No solutions

2. a) 90^0

b) 30^0, 12^0.

c) 3.99+ 2**π**n, 5.444 + 2**π**

d) 12^0+260^0k, 24^0+36^0k

e) -$\frac{-2\pi}{3}$

f) -120^0, 60^0

g) -60^0, 120^0

h) -60^0, 60^0

i) $-135^0, 45^0$

3. a) No solution

$b)\ \frac{\pi}{3}, \frac{4}{3}\boldsymbol{\pi}$

$c)\ \frac{-\pi}{4}, \frac{3\pi}{4}$

$d)\ \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{-7\pi}{4}, \frac{-\pi}{12}, \frac{\pi}{12}$

$e)\ \frac{-\pi}{2}$

f) $45^0, 135^0$

b) $135^0, 225^0$

c) $30^0, 210^0$

d) 180^0

Solving quadratic trigonometric equations

Graphical solutions of Quadratic Trigonometric Equations

Exercise 2.2

Solutions

1. $30^0, 90^0, 270^0, 330^0$.
2. No solution
3. $135^0, 225^0$
4. $0^0, 120^0, 180^0, 240^0$
5. $30^0, 150^0, 210^0, 330^0$
6. $\frac{3\pi}{2}$
7. 63.4, 161^0, 243, 341

Using trigonometric identities in solving equations with more than one trigonometric function

Task

Solve the equation $\cos 2\theta + 2\cos^2\theta = 2$, $0° \le \theta \le 360^0$

Solution

Use double angle formula to remove (2θ)in the trigonometric.

Consider identity

$$cos\ 2\theta = cos^2\theta - sin^2\theta.$$

Substituting back

$$\cos^2\theta - sin^2\theta + 2\,cos^2\theta = 2.$$

$$3\,cos^2\theta - sin^2\theta = 2.$$

To have some trigonometric ratio we need to use Pythagoras identity

$$\sin^2\theta + \cos^2\theta = 1.$$

$$\sin^2\theta = 1 - \cos^2\theta.$$

Substituting basic we get

$$3\cos^2\theta - (1 - \cos^2\theta) = 2.$$

$$3\cos^2\theta - 1 + \cos^2\theta = 2.$$

$$4\cos^2\theta = 3.$$

$$\cos^2\theta = \frac{3}{4}.$$

Taking square root on both sides

$$\cos\theta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

$\theta = \cos^{-1} \ \pm\frac{\sqrt{3}}{2}.$

$\theta = 30^0, 150°, 210°, 330°$

Hence, $\theta = 30^0, 150^0 \ , 210^0 and\ 330^0$

Exercise 2.3

Solutions

1. $-135^0, -45^0, 45^0, 135^0$
2. $-135^0, -45^0, 45^0, 135^0$
3. $120^0, 180^0, 240^0$
4. $30^0, 150^0, 270^0$
5. $111^0, 249^0$
6. $48^0, 312^0$
7. $60^0, 180^0, 360^0$
8. $0^0, 60^0, 180^0, 300^0$
9. $0^0, 90^0$
10. a) d=90sinθ

    d=60sin (90-θ)

    b) 80.53

    88.77

11. a) i) sin$\theta = \frac{y}{x}$

    ii) Sin2$\theta = 1.75\frac{y}{x}$

    b) 28.94^0

14. a) i) 0.3m

    ii) 108m

    b) i) 0^0

    ii) 60^0

UNIT 3

CALCULUS 3

Additional Math Secondary 4	**Unit 3:Calculus**
Learn about	**Key inquiry questions**
Learners should determine and apply the product of two functions, quotients and implicit functions to find solution to problems through discussion. They should investigate the derivatives of trigonometric functions: sinx, cosx and tanx and use them to solve problems individually and in groups. Learners should determine and apply integration to find the area under the curve through demonstration and discussion. They should determine the integrals of powers of linear functions $ax + b$ and integration of trigonometric functions by referring to worked examples to be able to investigate and apply the concepts of integration to solve problems individually and in groups. They should supplement their learning by using the internet and working with others.	• How can we use the concepts of differentiation of products, quotients to solve problems? • Why do we call some function, implicit functions? • How can we differentiate trigonometry functions? • How can we use differentiation of product, quotient, implicit and trigonometric functions to solve real life problems? • How can we use the concept of integration to find the area under a curve? • How do we integrate the linear function $ax + b$ raised to a certain power? • In what way can we find the definite integral of trigonometric functions? • How can we apply integration of trigonometric functions in solving problems?

Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**
• Differentiation of product of two functions, quotient and implicit function • Differential of trigonometric functions: sinx, cosx and tanx • Applications of integration such as finding area under the curve. Integration of powers of linear function $ax + b$ and trigonometric functions	• Calculate the differentiation of product of two function, quotient and implicit function • Differentiate trigonometric function; sinx, cosx and tanx • Use differentiation of product, quotient, implicit and trigonometric functions to solve real life problems • Investigate the use of trigonometric identities in differentiating trigonometric functions • Use the concepts of definite integral to find area under curve • Use further techniques to integrate powers of liner functions $ax + b$ • Derive formulae for integration of trigonometric functions • Apply integration of trigonometric functions in solving real life problems. • Investigate the use of trigonometric identities in integrating trigonometric function.	• Build interest in calculus so as to shape future career • Appreciate the use of differentiation • Teamwork • Build interest in calculus so as to shape future career • Commitment and resilience to solve problems • Appreciate the use of integration of trigonometric function in real life situation • Work with others to innovate

Contribution to the competencies: Critical thinking: Analyse problems and develop solutions using differentiation and suggest and develop solutions to problems involving integration Cooperation: Work collaboratively with others towards solutions to difficult questions. Be tolerant of others and respectful of differing views when solving problems related to integration with colleagues Communication: Communicate the knowledge of differentiations coherently and clearly Communicate ideas and information coherently in problems related to integration
Links to other subjects: Physics: acceleration, velocity and displacement, trajectory Chemistry: heat enthalpy Geography: increase and decrease of population

Learning/ Teaching Materials

A computer, smart phone, graph book and graph board.

Learning/Teaching Activities

1. Guide learners in review of concepts discussed in calculus 1 and 2 in book 3 through questioning technique.
2. Derivation of general formula of differentiation by product rule and quotient rule is done.
3. Learners are guided on problem solving discussing on the product and quotient rule guided by the students' book.

4. Guide learners on review of implicit functions through questioning technique.
5. Class discussion and problem solving on differentiating implicit functions using example and exercise on students' book.
6. Guide learners on deriving derivatives of trigonometric function and on solving differential questions on as guided in students book.
7. Learners are guided using a chart on summary trigonometric functions derivatives as they also appear on mathematical table.
8. A case study on real life problems involving differentiation of trigonometric function as illustrated in exercise.
9. Discussion on integration by part is reviewed. This time round introduce problems that has differentiation of trigonometric functions.
10. Learners are guided on working out area under a curve by integration as illustrated in students' book.
11. Learners are guided into a debate and discussion of different methods of determining area under a curve such discussion to involve groups under:-
    i. Trapezoidal rule
    ii. Rectangular rule
    iii. Counting of boxes
    iv. Integration
12. Drawing of graphs and determining area under a curve and defined region as illustrated in students book.
13. Guide learners into a discussion that involve integrating powers of linear functions, $(ax + b)^n$, where a, b and n ate integers.
14. Discussion on integration of trigonometric functions being guided by student's book example.

15. Guide learner's in solving real life problems on integration using examples and questions in exercise 3.6.

Assessment

Learners should be able to solve problems in the student's book.

The product rule

The product rule is derived as follows

Prove:

If function f is differentiable the derivative of f is

$$f^1(x) = \lim_{h\to 0}\left(\frac{f\,(x+h)-f(x)}{h}\right) \quad .$$

$$(fg)^1(x) = \lim_{h\to 0}\left(\frac{fg\,(x+h)-fg\,(x)}{h}\right) \quad .$$

Since; $fg(x) \ = f(x)\ g(x)$

$fg(x\ +\ h) = f(x+h)\ g(x+h)$and hence

$$(fg)^1x = \lim_{h\to 0}\left(\frac{f\,(x+h)g(x+h) - f\,(x)g(x)}{h}\right)$$

Geometrically analyzing the numerator

Consider a rectangle with length = g (x) and width = f(x)

When a small increment to x is made such that x becomes $(x+h)$ the new length = $g(x+h)$ and new width = $f(x+h)$. The increment in dimensions results to increase in area by thee parts A,B and C as shown in figure 3.1 below

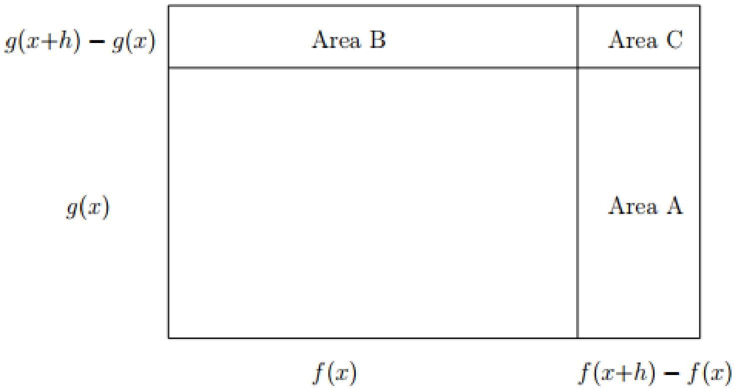


Figure 3.1

The area of the small rectangle $= f(x)\, g(x)$

The area of big rectangle $= f(x+h)\, g(x+h)$

The increased area $A + B + C = f(x+h)\, g(x+h) - f(x)\, g(x)$

Since the limit expression is similar to increase in area it can hence be expressed as summation of area A,B and C.

$$Area\ of\ A = (f(x+h) - f(x))\, g(x)$$

$$Area\ of\ B = (g(x+h) - g(x))\, f(x)$$

$$Area\ of\ C = (f(x+h) - f(x))\, (g(x+h) - g(x))$$

$$Area\ of\ A + B + C = (f(x+h) - f(x))\, gx + (g(x+h) - g(x))$$

Substituting this to the $\lim\limits_{h\to 0}$ and separating into the three parts we get

$$= \lim_{h\to 0} \frac{(\mathrm{f}(\mathrm{x+h}) - \mathrm{f}(\mathrm{x}) g(x))g(x)}{\mathrm{h}} + \lim_{h\to 0} \frac{(\mathrm{f}(\mathrm{x}) g(x+h)) - g(x))}{\mathrm{h}} + \lim_{h\to 0} \frac{(\mathrm{f}(\mathrm{x+h}) - f(x)) - g(x+h) - g(x))}{\mathrm{h}}$$

$$= g(x) \lim_{h \to 0} \frac{(f(x+h) - f(x))}{h} + \lim_{h \to 0} \frac{(f(x) g(x+h)) - g(x))}{h} + \lim_{h \to 0} \frac{(f(x+h) - f(x)) - g(x+h) - g(x))}{h}$$

$$= f'(x)\, g(x) + f(x)\, g'(x) + f'(x)\, .0$$

$$= f'(x)\, g(x) + f(x)\, g'(x)$$

Hence,

If $\quad (f(x)g(x))^1 = f'(x)\, g(x) + f'(x)\, g(x)$

The quotient rule

This rule is derives as follows.

<u>Prove</u>

Quotient rule is derived from product rule. Product rule states,

If $\quad y = uv\,, \quad \frac{dy}{dx} = \frac{dy}{dx} v + u \frac{dy}{dx}$

Then,

If $\;y = \frac{u}{v}\;,\; y = u\,.\frac{1}{v}\;,$

$$\frac{dy}{dx} = \frac{\frac{dy}{dx}\mathrm{v}}{v} + u\frac{1}{v}\left(\frac{-dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$

Or $\quad \frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}\right)$

ANSWERS

Task

The function $f(x) = x^4$ can be considered as a product of two separate functions: $u(x)$ and $v(x)$.

$$f'(x) = 4x^3$$

Copy and complete the table below and discuss a suitable combination of u, v and their derivatives u', v' that will always be equal to $f'(x)$

$u(x)$	$v(x)$	$u'(x)$	$v'(x)$
1	x^4	0	$4x^3$
x			
x^2			
x^3			
x^4			
x^5			
x^6			

Differentiation by the product and quotient rule

Task

Use the product rule to derive the quotient rule.

Hint: replace v with v^{-1}

Exercise 3.1

Solutions

1. Differentiate
   a) $\frac{x^2(24-7x)}{2(4-x)^{\frac{1}{2}}}$
   b) $2x^5(1+\mathrm{x})^4(6+11\mathrm{x})$
   c) $-x^{-3}(1+\mathrm{x}^2)^{\frac{-1}{2}}(2+\mathrm{x}^2)$
   d) $\frac{1}{(x+1)^2}$

2. Find the derivative
   a) $\frac{-11}{(2x-4)^2}$
   b) $\frac{11}{(2x+1)^2}$
   c) $\frac{2(x^2+x+13)}{(2x+1)^2}$
   d) $\frac{-2(x^2+x+3)}{(x^2-3)^2}$
3. $\frac{-1}{3}$
4. $\frac{2(x^2-7x-6)}{(2x-7)^2}$

Exercise 3.2

Solutions

1. Differentiate the following implicit functions with respect to x for questions 1→10.
   a) $\frac{2y-2x+1}{2y-2x+1}$
   b) $\frac{2xy^3+3x^2y^2}{1-3x^2y^2-2x^yy}$
   c) $y^1 = \frac{\sqrt{x^2+y^2}-x}{y}$

d) $y^1 = \frac{1-y-3x^2-4x}{3y^2+x+2}$

e) $4x/(3y^2+2)$

f) $y^1 = \frac{5x^4y^7-4x^3}{4y^3-7x^5y^6}$

g) $6x^2/(10y+5)$

h) $y^1 = \frac{16xy^2-6x(x^2+y^2)^2}{6y(x^2+y^2)^2-16x^2y}$

i) $3x^2/2y$

j) $\frac{-x}{y}$

2. $y = \frac{7}{6}x + \frac{13}{6}$

Derivative of exponential and logarithmic functions

Task

Sketch the graph of $f(x) = a^x$.

Use a graph plotter to investigate the derivative of $f(x) = a^x$, for different values of a.

Is there a value of a, for which $f(x) = f'(x)$?

In general, $f(x) = a^x$, $f'(x) = a^x \ln x$

when $a = e$, $f(x) = f'(x)$

Task

Find the gradient function for $y = \ln x$

Hint: If $y = \ln x$ then $x = e^y$

Derivatives of trigonometric functions

Task

In groups, prove the derivatives of the trigonometric functions in table 3.1.

$f(x)$	$f'(x)$
$tan\, x$	$\sec^2 x$
$cot\, x$	$-\csc x^2$
$csc\, x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$

Table 3.1

Task

i. Find the derivative of $y = \sin 2x$.

ii. Generalise to $y = \sin ax$ and $y = \cos ax$?

Hint: Make a substitution and use the chain rule

Solution

Let u = 2x

$$\frac{d}{dx} = 2.$$

$$y = \sin u.$$

$$\frac{dy}{du} = \cos u.$$

Using chain rule.

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2\cos u = 2\cos 2x.$$

You note that:

$$\frac{dy}{dx}(\sin ax) = a\cos ax$$

$$\frac{dy}{dx}\cos ax = -\sin ax$$

Exercise 3.3

Solutions

1. Determine the derivative
   a) $y' = \mathrm{x}(\mathrm{x}\cos \mathrm{x} + 2\sin \mathrm{x})$
   b) $y' = 12\sin 3\mathrm{x}$
   c) $y' = \mathrm{x}^2(3\cos \mathrm{x} - \mathrm{x}\sin \mathrm{x})$
   d) $y' = -2\sin \mathrm{x}\sin 2\mathrm{x} + \cos \mathrm{x}\cos 2\mathrm{x}$
   e) $y' = \frac{(\mathrm{x}+1)\cos \mathrm{x} - \sin \mathrm{x}}{(\mathrm{x}+1)^2}$
   f) $y' = \frac{2}{\cos^2(2\mathrm{x})}$
   g) $y' = 3(\mathrm{x}+1)\cos 3\mathrm{x} + \sin 3\mathrm{x}$
   h) $y' = \frac{9\mathrm{x} - 4\mathrm{y}^3\cos 2\mathrm{x}^2\mathrm{y}^3}{6\mathrm{xy}\cos 2\mathrm{x}^3\mathrm{y}^3}$
2. Figure 3.3
   a) $\frac{40}{x} = \sin \theta$
   b) $\frac{dx}{d\theta} = 40(\frac{-\cos\theta}{sin^2\theta})$

Integration

Task

In groups,

i. Sketch the graph of $f(x) = 4x - x^2$
ii. Find the area bounded by $f(x) = 4x - x^2$, and x-axis.

Solution

The area is shown in *Figure 3.5*

Figure 3.5

The x-intercepts are when $4x - x^2 = 0$

Factorising $x(4 - x) = 0$

$x = 0$ or 4

The area between f and $g = \int_a^b (f(x) - g(x))\, dx$

$$f(x) = 4x - x^2$$

$$g(x) = 0$$

$$f(x) - g(x) = 4x - x^2$$

$\mathrm{A} = \int_0^4 4x - x^2 dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \left(2 \times 4^2 - \frac{4^3}{3}\right) - \left(2 \times 0^2 - \frac{0^3}{3}\right) = 32 - \frac{64}{3}$

$= \frac{32}{3} = 10\ \frac{2}{3}$ square units

Task 2

In groups of four students,

i. Sketch the graph of $f(x) = x^2 - 2$ and $y = x$ on the same Cartesian plane.

ii. Show that the area bounded by the curve $y = x^2 - 2$ and line $y = x$ is $4\frac{1}{2}$ square units

Solution

The graphs are shown in figure 3.6 below.

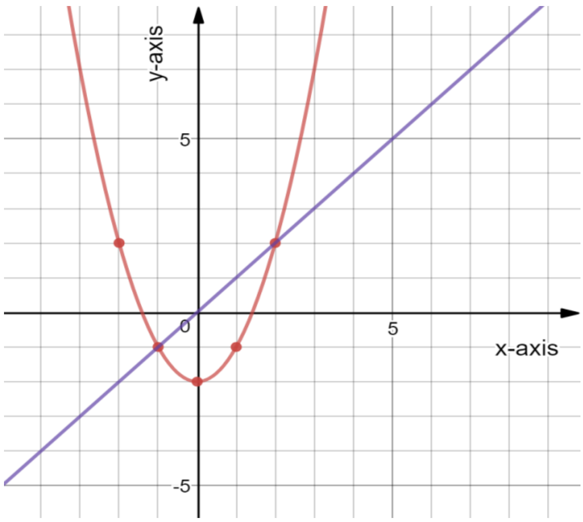


Figure 3.6

For area between f and g= $\int_a^b (f\ (x) - g(x))dx$ against Dx at { -a, b}

$$f(x) = x$$

$$g(x) = x^2 - 2$$

$f\ (x) - g\ (x) = x - (x^2 - 2) = x - x^2 + 2 = -x^2 + x + 2$

$f\ (x)$- $g\ (x) = (2 - x)(x + 1) = 0$ when $x = 2, -1$

$$a = -1\ , b = 2$$

A= $\int_{-1}^{2}(-x^2 + x + 2)dx = \left[\frac{-x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2}$

A $= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = \frac{9}{2} = 4\frac{1}{2}$ square units

Exercise 3.4

Solutions

1. Calculate
   a) 1
   b) $\frac{37}{12}$
   c) $\sqrt{2} - 1$
   d) $\frac{1}{12}$
   e) $\frac{1}{2}$
   f) $\frac{9}{2}$
2. $\frac{32}{3}$
3. $\frac{9}{6}$
4. $\frac{157}{6}$
5. A) Calculate the area
   i. 27
   ii. $\frac{32}{3}$

They are equal

Integration of exponential and logarithmic functions

Task 1

In groups, prove that

i. $\int \frac{1}{x}\, dx = \ln|x| + c$

ii. $\int e^x dx = e^x + c$

iii. $\int a^x dx = \frac{a^x}{\ln|a|} + c$

Exercise 3.5

Solutions

1) $\frac{1}{6}(x+4)^6 + c$
2) $\frac{728}{3}$
3) $\frac{1}{4}(x-2)^4 + c$
4) $930\frac{1}{50}$
5) $\frac{1}{16}(2x+1)^8 + c$
6) 4
7) $find\ the\ integral$
   a) $\frac{-5}{3}(1-x^3)^{\frac{3}{2}} + c$
   b) $\frac{1}{2}(x^4+16)^{\frac{1}{2}} + c$
   c) 0.0707
   d) $\frac{-10}{9}(1-x^3)^{\frac{3}{2}} + c$

Integration of trigonometric functions

Task 1

Use $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$

to establish $\int \sin x \, dx$ and $\int \cos x \, dx$

Task 2

Prove that $\int \tan x \, dx = -\ln|\cos x| + c$

Solution

From identity $\tan x = \frac{\sin x}{\cos x}$, $\int \tan x \;\, dx = \int \int \frac{\sin x}{\cos x} dx$

$u = \; cos\; x, \frac{du}{dx} = -\sin x,\; dx = \frac{du}{-\sin x}.$

Substituting basis the value of dx and cos x=u

$$\int \frac{\sin x}{\cos x} dx = \int \sin x \,. \frac{du}{\sin x} = \int \frac{1}{-u} du = -\int \frac{1}{u} du$$

$$= \; -\text{In}|u| \; + \; c$$

$$= \; -\ln|cos\; x| + \; c$$

Task 3

Prove that $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

Hint: multiply both numerator and denominator by $(sec\; x \; + \; tan\; x)$

to enable a substitution.

Solution

To enable us make substitution multiply both numerator and denominator by $(sec\; x \; + \; tan\; x)$

Hence,

$$\int \sec x \, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

Using u-substitution $u = \ sec\ x\ +\ tan\ x$

$$\frac{du}{dx} = \sec x \tan x + sec^2 x = \sec x(\tan x + \sec x)$$

$$dx = \frac{du}{\sec x(\tan x + \sec x)}$$

Substituting back the u and dx

$$\int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec x\,.\,(\sec x + \tan x).\,du}{\sec x\,(\tan x + \tan x)\,.\,u} = \int \frac{du}{u}$$

$= \ln\ u\ + c$, substituting u.

$= \ln|\sec\ x + \tan\ x\ | + c$

Exercise 3.6

Solutions

1. $\frac{1}{3}\sin(3x + 4) + c$
2. $\frac{-1}{7}\cos(7x - 3)$
3. $1.382\ square\ units$
4. -cos $\sqrt{x} + c$
5. $-\cos(1 - x^2) + c$
6. |n|1+sin $x| + c$
7. $\frac{-1}{4}\cos 2x^2$
8. $\frac{-1}{5+\sin x} + c$
9. $\frac{-1}{3}\cos 3x + c$
10. $x + \cos x + c$
11. $\frac{1}{5}|n|3 + \sin 5x| + c$
12. $\frac{5}{4}\sec 4x + c$

Application of integration to kinematics

Task

Use calculus to derive expressions for velocity (v) and displacement (s), for a particle moving at constant acceleration (a) with an initial velocity (u) and 0 initial displacement.

This task leads to the equations met in Additional Secondary 3: $v = u + at$ and $s = ut + \frac{1}{2}at^2$

Exercise 3.7

Solutions

1. a) 10.1 sec to 1 d.p.

b) 99.0m/s to 1 d.p.

2. a) $s = t(t-3)^2$, so at starting position after 3 seconds

b) acceleration is $6t - 12$. When $t = 3$, a =6 m/s^2

UNIT 4

PARTIAL FRACTIONS

<table>
<tr><td colspan="2">Additional Math   Secondary 4</td><td>Unit 4:Partial Fractions</td></tr>
<tr><td colspan="2">Learn about</td><td>Key inquiry questions</td></tr>
<tr><td colspan="2">Learners should be introduced to partial fractions and identities through discussion on how partial fractions which have a denominator with linear factors, a denominator with irreducible quadratic factors and a denominator with repeated factors, are determined.

They should investigate the use of partial fractions in solving problems in mathematics and other subjects. They should supplement their learning by using internet, reference textbooks and working with others.</td><td>• How can we distinguish between equations and identities?
• In what way can we find the value of constants of identities?
• Why is it important to classify partial fractions?
• How can we use the concepts of partial fraction in problems solving?
• How do we separate functions into partial fractions?</td></tr>
<tr><td colspan="3" align="center">Learning outcomes</td></tr>
<tr><td>Knowledge and understanding</td><td>Skills</td><td>Attitudes</td></tr>
</table>

• Introduction, identify denomination or with only linear factors, with quadratic factors and with repeated factors • Vectors in terms of i, j and k, application of vector method in geometry	• Use different techniques to classify partial fractions. • Express partial fractions as a single fraction. • Solve problems involving partial fractions in real life situations. • Investigate the use of partial fraction in solving real life problems.	• Appreciate the use of partial fractions in solving problems in mathematics • Value the importance of partial fractions in sciences. • Teamwork
Contribution to the competencies: Critical thinking: Analyse critically the different classifications of partial fractions Cooperation: Develop solutions to problems using partial fractions in groups		
Links to other subjects: Chemistry: Express the relationship between product pressures and reactant pressures Physics: Dalton's law ICT: Internet		

Learning/Teaching Activities

Introduction

1. Defining partial fraction

2. Discussion on expressing improper algebraic fractions into a mixed number. The guide learners to understand the two techniques that can be applied;-
   i. Factorized simplification

Partial Fractions of linear factor(s) in Denominator

1. Discussion on identification of this kind of fractions.
2. Discussion on type of decomposition applied depending on nature of the denominator power or factors. Use visual illustrations and charts to enable learners to master this decomposition.
3. Equating general expression with co-efficient to the fraction and decomposition it by solving for the coefficients. Learners are guided on the process of solving the complex equation to obtain values of coefficients. Emphasize on;
   i. Comparing similar coefficients.
   ii. Substitution of x that eliminate the coefficient simplifying coefficient equation. Use of zero factors should be emphasized to solving simultaneous equations as shown in examples on students' book.

Partial Fractions of Quadratic Factor(S) In the Denominator.

1. Discussion on types of quadratic equations and existence or non-existence of their factors. Teacher guide learners review types of factors of linear equations.
2. Learners are guided on types of decomposition for quadratic denominators with and without factors. A chart on visual illustration is used.

3. Discussion on decomposition of such fraction using examples and exercise in student's book.

Application of Partial Fractions

1. Teacher guide learners to review the concept of integration. He illustrates some integrals that were unsolvable using previous learnt method through review of previous visit.
2. Learners are guided on how partial fractions convert an integratable algebraic fraction into ones that can be solved using u-substitution. Emphasize on the technique used using examples in students' book.
3. Discussion on real life problems that require partial fraction integrations are discussed by the guidance of examples and questions in learners book.

Expressing Partial Fractions as a single Fraction

Teacher review on ability of learners t express partial fractions as a single fraction by carrying out the process of adding process that involve;

i. Identifying the L.C.M
ii. Multiply each fraction by the L.C.M and add or subtract then divide by it.

Assessment

Teacher guide learner through all the examples. He also ensure they can solve all the questions in the student's book.

ANSWERS

Task

In pairs

1. Simplify the expression $\frac{x^3}{x^2-1}$ into a mixed number.
2. Express the following fractions as a single fraction
   a. $\frac{1}{(x-1)^2}+\frac{4}{(x-1)^3}$

   b. $1+\frac{4}{x^4+1}$

Solution

Since the numerator has a greater power than the denominator the fraction has been decomposed into parts

Consider $\frac{x^3}{x^2-1}=\frac{x(x^2-1)+x}{x^2-1}$

You notice x^3 can be written as $x(x^2-1)+x$ while on expansion the added term is eliminated technically $x(x^2-1)-x=x^3-x+x=x^3$

To Simplify $\frac{x^3}{x^2-1}=\frac{x(x^2-1)+x}{x^2-1}=\frac{x(x^2-1)}{x^2-1}+\frac{x}{x^2-1}=x+\frac{x}{x^2-1}$

$$\frac{x^3}{x^2-1}=x+\frac{x}{x^2-1}$$

Exercise 4.1

Solutions

1. x^2+3x+4

2. $x^2 - 24x + 26 - \frac{25x-56}{x2+x-2}$
3. $3x + 1$
4. $3x + 2 + 4$
5. $\frac{3}{2}x^2y - \frac{5}{2} x^2$

Partial Fractions with linear factor(s) in the denominator

Exercise 4.2

Solutions

1. $\frac{5}{8(x+5)} + \frac{3}{8(x-3)}$
2. $\frac{1}{(x-1)^2} + \frac{4}{(x-1)^2}$
3. $\frac{-1}{5(x+2)} + \frac{1}{(x+2)^2} + \frac{1}{5(x-3)}$
4. $\frac{1}{9(x+1)} + \frac{10}{9(x-5)^2} + \frac{26}{9(x-5)}$
5. $\frac{-5}{9(x+2)} - \frac{8}{9(x+2)} + \frac{22}{9(x-4)}$
6. $6 + \frac{3}{(x-2)} - \frac{18}{(x+4)}$

Partial Fractions of functions with a quadratic expression in the denominator

Exercise 4.3

Solutions

1. $x + \frac{x}{(x^2-1)}$
2. $\frac{3}{4(x+3)} + \frac{1}{(x-1)}$
3. $\frac{5}{8(x+5)} + \frac{3}{8(x-3)}$

4. $\frac{2}{27(x-2)} - \frac{2}{27(x+1)} + \frac{7}{9(x+2)^2} + \frac{1}{3(x+1)^3}$
5. $\frac{x}{x^2+1} - \frac{x}{x^2+2}$
6. $\frac{1}{27((x-1^2)} - \frac{1}{27(x+1)} + \frac{1}{27(x^2+x+1)} - \frac{1}{9(x^2+x+1)^{\,2}} - \frac{1}{3(x^2+x+1)^{\,3}}$
7. $\frac{x}{x^2+1} - \frac{x}{x^2+2}$
8. $\frac{1}{3(x-1)} + \frac{1}{4(x+1)} + \frac{1}{2(x+5)}$
9. $\frac{x+1}{x^2+X+4} - \frac{1}{X+2}$
10. $\frac{2}{X-3} - \frac{1}{X+2}$
11. $\frac{8x}{x^2+2} - \frac{3}{(x^2+2)^2}$
12. $\frac{1}{X} + \frac{2x+4}{x^2+4}$
13. $\frac{51}{40(x-6)} - \frac{11x+14}{40(x^2+4)}$
14. $\frac{-25}{17(2x-3)} + \frac{37}{17(5x-1)}$

Application of partial fractions in integration

Exercise 4.4

Solutions

1. $2\ln x + ln\,|3\mathrm{x}+4| - \; 8ln|\mathrm{x}+1|$ +c
2. $\left|\frac{1}{9}\ln 5 - \ln 9 + \ln\frac{79}{9}\right|$
3. $\frac{37}{85}|5\mathrm{x}-1| - \frac{25}{35}ln|2\mathrm{x}+3| + \; c$
4. $\frac{1}{-6}ln|\mathrm{x}+1| - \frac{1}{3}ln|\mathrm{x}-2| + \frac{2}{3}ln|\mathrm{x}-5| + c$

UNIT 5

VECTORS

<table>
<tr><td colspan="2">Additional Math Secondary 4</td><td colspan="2">Unit 5:Vectors and Partial Fractions</td></tr>
<tr><td colspan="3">Learn about</td><td>Key inquiry questions</td></tr>
<tr><td colspan="3">The learner should learn through demonstration by the teacher and discussion between the teacher and with them how to express partial fractions and vectors in terms of i, j and k, add and subtract vectors and geometrically find the position vector.

They should be able to apply vectors to find the equation of a line, the length of a line (Pythagoras theorem) and the scalar product of two vectors.

They should also investigate the distributive law and use it in solving problems involving vectors individually and in groups.

They should supplement their learning by using internet and working with others.</td><td>• How can vectors be added and subtracted in terms of i, j and k?
• How can we apply vectors in geometry and algebra?
• In what ways can we apply vectors to work out problems?</td></tr>
<tr><td colspan="4">Learning outcomes</td></tr>
<tr><td>Knowledge and understanding</td><td colspan="2">Skills</td><td>Attitudes</td></tr>
</table>

Learning/Teaching Materials

A ticker timer, Graph book, Graph board, a 3-dimentional method box for co-ordinate, chart, scientific calculator, geometrical set.

• Introduction to partial fractions., identify denomination or with only linear factors, with quadratic factors and with repeated factors Vectors in terms of i, j and k, application of vector method in geometry	• Add and subtract vectors in term of i, j and k • Solve vector equation of a line passing through two points • Use Pythagoras theorem to find the length of a vector • Find the scalar product of two vectors • Derive distributive law as applied in vectors • Work out problems involving vectors • Investigate the use of vector methods in geometry in everyday life	• Appreciate the use of vectors in solving problems. • Value the importance of vectors and its applications in other subjects. • Build confidence in determining problems related to vectors. • Show resilience in discussing vectors

Contribution to the competencies:

Critical thinking: Analyze and apply vector methods in evaluating problems

Cooperation: Be tolerant of others and respectful of differing views, when working together

Communication:Use appropriate means to communicate ideas of vectors

Links to other subjects:

Geography: taking direction

Physics: vector quantities

ICT: Internet

Learning/Teaching Activities

Introduction

1. Use auditory method to review the concepts of vector and scalar quantities as discussed in previous units. He guides learners to define scalar quantities and vector quantities.
2. Listing of vector quantities and scalar quantities.
3. Making visual illustration to represent vectors diagrammatically.
4. Discussion on vector notation and types of vectors is also reviewed.
5. Drawing visual illustration for three dimensional geometry and Cartesian plane are made.

Positional Vector in 3-Dimensions

1. Learners are guided using a 3-D Cartesian plane (the meshed box) on determining coordinates in these dimensions.
2. Demonstration is done using visual illustrations in figure 5.3 of student's book.
3. Discussion on positional vector in 3-Dimension is made.
4. Guide learners on discussion in equal vectors and fee space vector description using figures 5.6.
5. Discussion on addition a vectors in 3-Dimension is done using visual literacy in figures 5.8 and 5.9 in students book.

Addition of vectors in Cartesian plane

1. Learners are guided on methods of adding vectors.

2. Vector addition are visually illustrated on a Cartesian plane in 3-Dimension. Teacher emphasis on technique applied in adding vector not in plane.
3. Addition and subtraction of vectors in ***i, j* and *k*** notation is done.

Magnitude of Vector

1. Discussion through questioning technique is used to review the magnitude of vectors in 2-Dimension.
2. Using visual aids learners are guided on magnitude of vector in 3D. The relation obtained is used to guide learners to guide learners to determine magnitude of vectors without making visual illustrations as guided in the students' book.

Scalar Multiplication of Vector

1. Guide learners in review of scalar multiplication of a vector.
2. Discussion on distributive law of vector, cumulative law and associative law are done. Teacher guide students on usage of this law and their application using the students' book.

Vector Equation of a line

1. Using visual illustration teacher guide learners through the process of determining and expressing the vector equation of a line.
2. Derivation of a general vector equation of a line is done.
3. Discussion on writing vector equation of lines is done using examples in students' book.

Cartesian equation of a straight line.

1. Discussion on derivation of Cartesian equation of a line is done. Teacher guide the learners on the relationship of a vector equation, column vector equation parametric equations and Cartesian equation of straight line.
2. Discussion on determining Cartesian equation of straight line passing through give points in 3-Dimensios.
3. Guide learners on use of a general equation in writing Cartesian equations using examples or students book.

The Scalar Product (The dot product)

1. Discussion on those products of vectors is done as provided in the student’s book.
2. Learners are guided through the visual illustration and its definition as provided in student’s book.
3. Discussion on solving problems using scalar products is done.
4. Discussion on properties of scalar properties are done. Emphases are made on cumulative and distributive law of vectors and their application to scalar product.
5. Application of scalar product to orthogonal vectors is discussed. Teacher guide learners in defining orthogonal vectors. Using a chart on visual illustration of orthogonal vectors. Guide learners on these vectors and practically illustrate orthogonal vectors in real life scenarios.

Scalar products of two vectors on a Cartesian plane

1. A discussion is made guided by the teacher for learners to understand the process of multiplying by scalar (dot) two vectors.

2. Derivation of a general scalar product is done.
3. Discussion on process of scalar multiplying vectors. Teacher emphasis use of column vectors in those processes.

Application of scalar product

- Discussion on application of scalar product to perpendicular vectors (orthogonal) vectors is done.
- Drawing of illustration orthogonal vectors. Problem solving on orthogonal vectors.
- Finding the angle between two vectors using scalar product is done.
- Determining component of a vector interim of another is discussed as illustrated in students' book.
- Definition of a unique vector
- Solving real life problems involving vectors is done guided by the students' book.

Assessment

The learners should be able to make clear illustrations of vectors and draw these vectors in a graph. They should be able to solve questions in the exercises provided in students' book. Teacher should guide student in area they encounter challenges.

ANSWERS

Task 1

1. List at least three examples of vector quantities.
2. List least three examples of scalar quantities.

Position vectors in 3 dimensions

Task 2

In pairs, study figure 5.4 and answer the questions that follow.

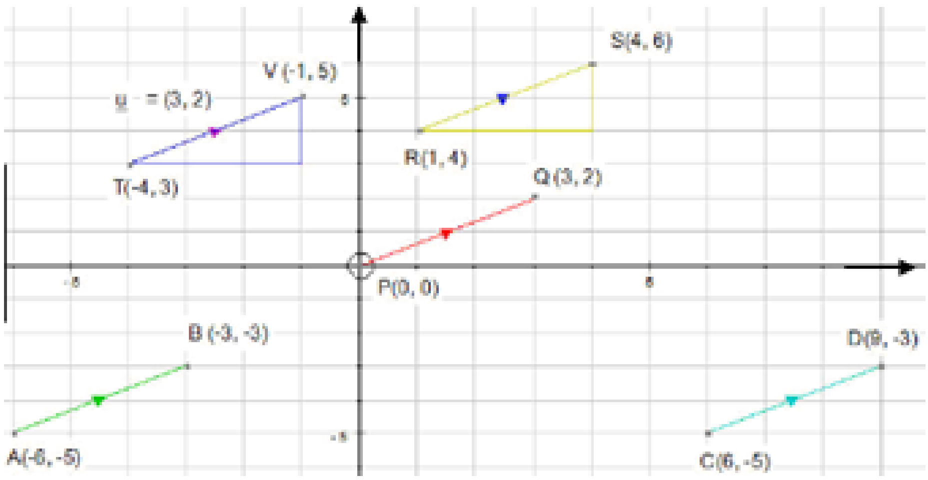


Figure 5.4

a. Determine the column vectors for $\overline{AB}, \overline{CD}, \overline{PQ}, \overline{TV}$ and $\overline{RS}$
b. Write the vectors $\overline{AB}, \overline{CD}, \overline{PQ}, \overline{TV}$ and $\overline{RS}$ in terms of unit vectors ***i, j*** and ***k*** form
c. What do you notice about these vectors? The vectors are all equal $\begin{pmatrix}3\\2\end{pmatrix} = 3\boldsymbol{i} + 2\boldsymbol{j}$

Addition of vectors

Task 1

In pairs, study figure 5.6 below and answer the questions that follow.

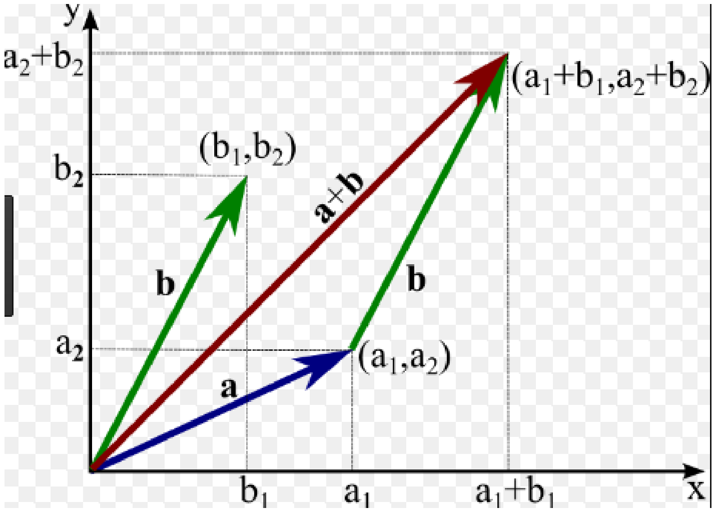


Figure 5.6

a. Write vector $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ as a column vector in terms of a_1, b_1, a_2 and b_2.

b. If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ find the value of **a** + **b**

Solution

To get **a** + **b** move **b** such that its tail contact the head of **a** .You notice if

$$\mathbf{a} = \mathbf{OA} = \begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix}$$

$$\mathbf{b} = \mathbf{OB} = \begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{pmatrix}$$

Then, $\mathbf{a} + \mathbf{b} = \mathbf{OA} + \mathbf{OB} = \begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix} + \begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{pmatrix} = \begin{pmatrix}\mathbf{a}_1 + \mathbf{b}_1\\ \mathbf{a}_2 + \mathbf{b}_2\end{pmatrix}$

In unit vectors

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$

$\mathbf{b} = b_1 i + b_2 \mathbf{j}$, then

$\mathbf{a} + \mathbf{b} = a_1\mathbf{i} + b_1\mathbf{i} + a_2\mathbf{j} + b_2\mathbf{j}$

$\mathbf{a} + \mathbf{b} = (a_1+b_1)\mathbf{i} + (a_2+b_2)\mathbf{j}$

In 3 dimensions

If; $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$\mathbf{a}+\mathbf{b} = (a_1+b_1)\mathbf{i} + (a_2+b_2)\mathbf{j} + (a_3+b_3)\mathbf{k}$

Task 2

Given $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ find in pair,

a. $2\boldsymbol{a}$
b. $\boldsymbol{a} + \boldsymbol{b}$

Solution

a. $2\boldsymbol{a} = 2(2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) = \mathbf{4}\boldsymbol{i} + 6\boldsymbol{j} + 6\boldsymbol{k}$
b. $\boldsymbol{a} + \boldsymbol{b} = 2\boldsymbol{i} + \mathbf{3}\boldsymbol{i} + 3\boldsymbol{k} + 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k} = 4\boldsymbol{i} + 7\boldsymbol{j} + 4\boldsymbol{k}$

Exercise 5.1

Solutions

1. m=7

   b=5

2. $\mathbf{a} = 9\mathbf{i} + 12\mathbf{j}$

   $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$

3. a) $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

b) 2**j**+2**k**

c) 2i + j +3k

d) 8i + 12k

4. a) 4**i**+8**j**+**k**

b) 7i+9j+k

c) 9i+7j+2k

Magnitude of vector (Length of a line and vector)

Task

In groups of three, determine the length of the line OP in the figure 5.7 below given the coordinates of P are (2, 3, 5)

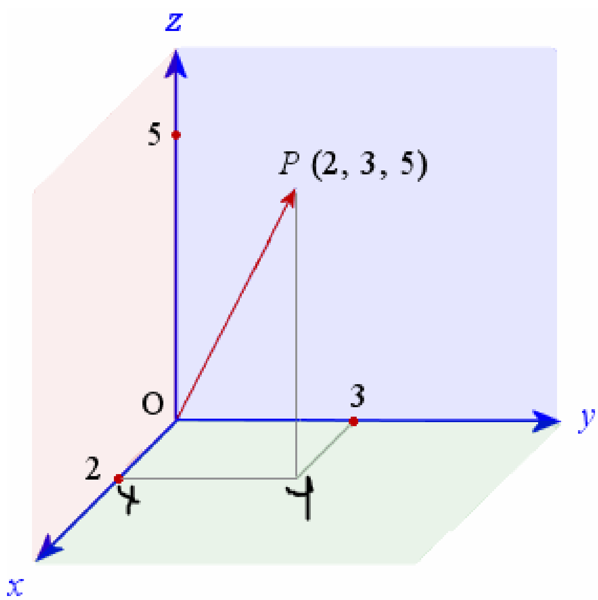


Figure 5.7

Solution

Using right triangle OXY and Pythagoras theorem

$$\mathbf{OY}^2 = 2^2 + 3^2$$

Using right triangle OYP and Pythagoras theorem

$$|\boldsymbol{OP}|^2 = |OY|^2 + |YP|^2$$

$$|\boldsymbol{OP}|^2 = 2^2 + 3^2 + 5^2$$

$$|\boldsymbol{OP}| = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$$

You notice that for P(2, 3, 5), $\overline{OP} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $|\overline{OP}| = \sqrt{2^2 + 3^2 + 5^2}$

In general, for a vector $= \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, its length/magnitude is

$$|\boldsymbol{a}| = \sqrt{(x^2 + y^2 + z^2)}$$

Task

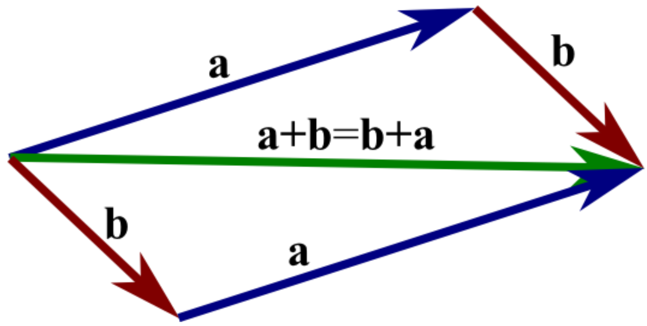
In groups, establish the laws for arithmetic of vectors:

1. The commutative law $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
2. The associative law $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$
3. The distributive $k(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = k\boldsymbol{a} + k\boldsymbol{b} + k\boldsymbol{c}$

Hint: use diagrams in each case.

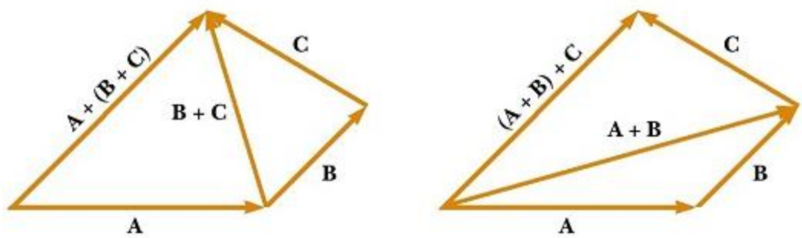
Commutative law of vector

States that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$



Associative law of vector

States that $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$



The unit vector

Exercise 5.2

Solutions

1. a) 2i-3j-5k

   b) -2i+3j-5k

   c) 4i+2k

   d) $\sqrt{38}$

   e) $\frac{1}{\sqrt{38}}(2i + 3\boldsymbol{j} - 5k)$

2. $\frac{1}{\sqrt{22}}(2i = 3j + 3k)$

3. a) 4k

b) 8i + 6j + 14k

c) 3**j** + **k**

The vector equation of line

Task 1

In groups, show that a vector equation for the line passing through the points whose position vectors are $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = 7\boldsymbol{i} + 5\boldsymbol{j}$ is

$$\boldsymbol{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + k\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Solution

Consider **general case**. The expression $\boldsymbol{p} = \boldsymbol{a} + k[\boldsymbol{b} - \boldsymbol{a}]$ is called **vector equation** of a line through point A and B with position vectors **a** and **b** respectively.

$$\boldsymbol{p} = \boldsymbol{a} + k[\boldsymbol{b} - \boldsymbol{a}]$$

$$\boldsymbol{p} = 3\boldsymbol{i} + 2\boldsymbol{j} + k[7\boldsymbol{i} - 5\boldsymbol{j}] - [3\boldsymbol{i} = 2\boldsymbol{j}]$$

$$\boldsymbol{p} = 3\boldsymbol{i} + 2\boldsymbol{j} + k([7\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{j} - 2\boldsymbol{j}]$$

$$\boldsymbol{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + k\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Task 2

In groups, write a vector equation for the line passing through the points with position vector $\boldsymbol{a} = 5\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}\ and\ \boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{i} - 2\boldsymbol{k}$

Solution

$$\boldsymbol{p} = \boldsymbol{a} + k[\boldsymbol{b} - \boldsymbol{a}]$$

$$\boldsymbol{p} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} + k\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

$$\boldsymbol{p} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} + k\begin{pmatrix} -3 \\ 5 \\ -5 \end{pmatrix}$$

Exercise 5.3

Solutions

1) a) $p = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} + k\begin{pmatrix} +3 \\ -3 \\ +7 \end{pmatrix}$

b) $q = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + k\begin{pmatrix} -2 \\ -3 \\ -2 \end{pmatrix}$

c) $\boldsymbol{p} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + k\begin{pmatrix} +2 \\ +1 \\ +2 \end{pmatrix}$

2. i)$\boldsymbol{p} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + k\begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}$

ii) $\boldsymbol{e} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + k\begin{pmatrix} -3 \\ -4 \\ -6 \end{pmatrix}$

iii) $\boldsymbol{p} = \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix} + k\begin{pmatrix} -8 \\ -1 \\ -10 \end{pmatrix}$

Cartesian form of the equation of straight line

Task

i. Show that the Cartesian equation of the straight line A (5, 6, 7) and B (1, 2, 2) is $\frac{x-5}{4} = \frac{y-6}{4} = \frac{z-7}{5}$

ii. State the equivalent vector equation.

Solution

Cartesian equation of a straight line passing through points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

The Cartesian equation is found by substituting in

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$\frac{x-5}{-4} = \frac{y-6}{-4} = \frac{z-7}{-5}, \frac{x-5}{4} = \frac{y-6}{4} = \frac{z-7}{5}$

To obtain the vector equation $\boldsymbol{p} = \boldsymbol{a} + k(\boldsymbol{b} + \boldsymbol{a})$, consider the general column vector equation

Substituting into this equation we get

$$\boldsymbol{p} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + k\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + k\begin{pmatrix} -4 \\ -5 \\ -5 \end{pmatrix}$$

Exercise 5.4

Solutions

1) a) $\frac{x-9}{-5} = \frac{y-3}{=2} = \frac{2-2}{1}$

b) $p = \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix} + k \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$

2) $x = -2 + 3k$

$y = 1 + 2k$

$\boldsymbol{z} = 5k$

3) $\frac{x-1}{2} = \frac{y+2}{4} = \frac{z-2}{-4}$

4) a) i + j+ k

b) $p = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

The scalar product

Task 1

Calculate the angle between vector $\boldsymbol{a}$ whose magnitude is 4 and vector $\boldsymbol{b}$ whose magnitude is 5 with an angle between them of 60^0.

a. Calculate the scalar dot product
   i. $\boldsymbol{a.b}$
   ii. $\boldsymbol{b.a}$

b. what do you notice about $\boldsymbol{a.b}$ and $\boldsymbol{b.a}$?

Solution

ai.

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

$$\boldsymbol{a}.\boldsymbol{b} = 4 \times 5 \times \cos 60° = 10$$

aii.

$$\boldsymbol{b}.\boldsymbol{a} = |\boldsymbol{b}||\boldsymbol{a}|\cos\theta$$

$$\boldsymbol{b}.\boldsymbol{a} = 5 \times 4 \times \cos 60° = 10$$

Hence $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a} = 10$

Task 2

In pairs,
Establish that the scalar produce is distributive i.e.

$$\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$$

Investigate what happens to the scalar product if the angle between the vectors is 90°.

Scalar product of two vectors on Cartesian plane

Task

In groups

Copy and complete the table of scalar products

.	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$

i			
j			
k			

Hence, determine the value of $\boldsymbol{a.b}$ given that $\boldsymbol{a} = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}$ and $\boldsymbol{b} = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$

Solution

.	***i***	***j***	***k***
i	1	0	0
j	0	1	0
k	0	0	1

Applying this scenario to position vectors of general points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) we get

$$\mathbf{a.b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} . \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

Exercise 5.5

Solutions

1) 51
2) -116
3) a) 30

b) 30

c) 97

d) 13

4) 0

5) a) 27

b) 27

c) 29

d) 126

6) Both are 77

7) a) -50

b) -50

c) 29

d) 201

Application of scalar products

Finding the angle between two vectors

Task

Use

1) $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ and

2) $\boldsymbol{a}.\boldsymbol{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} . \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$

to find the angle θ between the vectors $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{k}$

Solution

Use the relationships

1) $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ and

2) $\boldsymbol{a}.\boldsymbol{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} . \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$

Substituting in 2

$$\boldsymbol{a}.\boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} . \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 2 \times 3 + 0 \times 3 + 4 \times 4 = 16 + 6 = 22$$

Substituting in 1

$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

$$|\boldsymbol{a}| = \left|\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\right| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$|\boldsymbol{b}| = \left|\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}\right| = \sqrt{3^2 + 4^2} = 5$

Substituting $\boldsymbol{a}.\boldsymbol{b} = 22$, $|\boldsymbol{a}| = \sqrt{29}$ and $|\boldsymbol{b}| = 5$ in $\mathbf{a}.\mathbf{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ we get

$$22 = 5\sqrt{29}\cos\theta$$

$$\cos\theta = \frac{22}{5\sqrt{29}}, \theta = \cos^{-1}\frac{22}{5\sqrt{29}} = 35.2° to\ 1 d.p.$$

Finding the component of a vector in terms of another

Task

In pairs, show that the component of $\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{i} + 3\boldsymbol{k}$ in the direction of $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ is $\frac{17\sqrt{14}}{14}$

Solution

Use component vector $l = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|}$

$$|\boldsymbol{a}| = \left|\begin{pmatrix}2\\3\\1\end{pmatrix}\right| = \sqrt{(22 + 32 + 12)} = \sqrt{14}$$

$$l = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix}.\begin{pmatrix}4\\2\\3\end{pmatrix} = \frac{\sqrt{14}}{14}(8 + 6 + 3) = \frac{17\sqrt{14}}{14}$$

Exercise 5.6

Solutions

1) They are
2) 27.9^0
3) $\sqrt[2]{3}$
4) Ai = 4, $\theta = 63.4°$
5) 1.987
6) Perpendicular
7) 131.8^0, 48.2^0

UNIT 6

COMPLEX NUMBERS

Learning/teaching Materials

Additional Math Secondary 4		**Unit 6:Complex Numbers**
Learn about		**Key inquiry questions**
Learners should find out how to represent complex numbers graphically and in polar form, explain properties of the polar form of complex numbers, the power and De-Moivre's theorem. They should learn about the roots of complex numbers and solution of quadratic equations in complex numbers through discussion between the teacher and with them. They should develop analytical thinking and start to understand quantum mechanics and wave theory, and learn how to investigate complex numbers and use them in solving problems. They should supplement their learning by using the internet, reference textbooks and working with others.		• How can we represent complex numbers graphically and in polar form? • How can we derive some properties of complex numbers in polar form? • In what ways can we apply De Moivre's theorem in solving problems in complex numbers? • How can we evaluate roots of complex numbers? • How can we solve quadratic equations with complex numbers?
Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**

A graph book, a graph board, geometrical set, scientific calculator, computer.

• Graphical representation and polar form of complex numbers • The powers and De-Moivre's theorem • The roots of complex numbers and solution of quadratic equation in complex numbers	• Represent complex numbers graphically and in polar form. • Work out some properties of the polar form of complex numbers. • Find roots of complex numbers. • Solve quadratic equations in complex numbers. • Work out problem involving complex numbers using De Moivre's theorem.	• Appreciate the ideas behind complex numbers and their use and representation. • Generate ideas and value the concepts of complex numbers in real life situations. • Team work

Contribution to the competencies:

Critical thinking: Analyze and develop solutions to problems using complex numbers

Communication: Communicate the ideas of complex numbers clearly and effectively to others

Cooperation: Investigate the use and application of complex numbers in other subjects in groups

Links to other subjects:

Physics: Quantum mechanics, electromagnetic waves

ICT: Internet

Learning/ Teaching Activities

Introduction

1. Learners are guided on review of complex number unit studied in Additional Mathematics Secondary 3.
2. Discussion and emphasis are made on a complex number being a composition of a real number and imaginary number.

Graphical Representation of complex number

1. Discussion and introduction of an imaginary Cartesian plane is done using visual illustrations made on x-axis having real values and y-axis being called imaginary axis and having imaginary numbers.
2. Definition of an argand diagram.
3. Drawing Argand diagram as illustrated in students book examples.
4. Discussion on drawing argand diagrams.

Polar form of a complex number

1. Definition of a polar equation is done.
2. Discussion on expressing polar equation is done using visual illustrations in student’s book. Make emphasis on the process of converting Cartesian equations to polar equation and their significance. The components of a polar equations and their significance is also emphasized. Learners are guided in problem solving as illustrated in learners book, examples and exercise 602.

De Molvre's Theorem

1. Learners are guided to derive and define the De Molvre's theorem as illustrated in students' book.
2. Discussion on using De Molvre's Theorem to derive trigonometric identities is done using guided questioning techniques to solve example and exercise 6.3 in students' book.
3. Learners are guided in discussion on how De Molvre's theorem is used to determine root of complex numbers using examples in students' book.

Solving quadratic Equations with complex solutions

1. Guide learners in reviewing solutions of quadratic equations.
2. A discussion of solving equations whose discriminant $\sqrt{b^2 - 4ac} < 0$ which were previously stated as insoluble are emphasized as having complex number solution as illustrated using examples in students book.

Assessment

Learners are guided to solve the questions in the examples and exercises so as to master the concepts and apply it in real life.

Answers

Polar form of a Complex number

Task

Show that the complex number $z = 2 + 2i$ can be written as

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\ \sin\frac{\pi}{4}\right)$$

Hint: sketch out the position of z to help determine θ

Solution

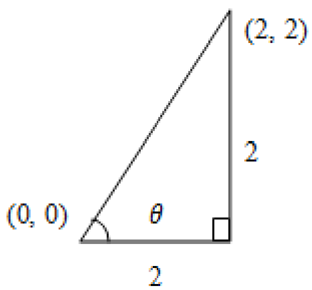
Consider general case: $z = x + yi$

Polar form: $z = r\ (\cos\theta + i\sin\theta)$

$r=\sqrt{x^2 + y^2}$

$r= \sqrt{2^2 + 2^2} = \sqrt{8} = \sqrt[2]{2}$

Using



$r=\sqrt{2^2 + 2^2}= \sqrt[2]{2}$ this is the modulus $\tan\theta=\frac{2}{2}$

$\theta = \tan^{-1} 1 = 45^0 = \frac{\pi}{4}$ this is the argument in the interval of $\left[0\,,\frac{\pi}{2}\right]$

The polar form is hence, $z = \sqrt[2]{2}\ (\cos\frac{\pi}{4} + i\ \sin\frac{\pi}{4})$

Exercise 6.1

Solutions

a. 2

b. -$\frac{\pi}{3}$

c. $z = 2(cos\frac{\pi}{3} + i\ sin\frac{\pi}{3})$

1. $z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\ \sin\frac{\pi}{4}\right)$

$$z_2 = 3\sqrt{2}\left(5\cos\frac{\pi}{4} + i\ \sin\frac{5\pi}{4}\right)$$

$$z_3 = 1\left(\cos\frac{3\pi}{2} + i\ \sin\frac{3\pi}{2}\right)$$

$$z_4 = \sqrt{10}(\cos 18^0 + i\ \sin 18^0)$$

$$z_5 = \sqrt{5}(\cos 63^0 + 63^0)$$

2. $z_3 = 10(\cos 53 + i\ \sin 53)$

3. $z_6 = 12\left(\cos\frac{\pi}{2} + i\ \sin\frac{\pi}{2}\right)$

Task

Given $z = (\cos\theta + i\ \sin\theta)$ use the addition triangle of binomial coefficients to find expressions for

a. z^2

b. z^3

c. z^4

Deduce a general expression for z^n

Hint: Use trigonometric identities e.g.

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 - 2\sin^2\theta = \cos 2\theta$$

$$2\ cos\theta \sin\theta = \sin 2\theta$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

Solution

a) $z = (\cos\theta + i\ \sin\theta)$

$z^2 = (\cos\theta + i\ \sin\theta)^2$

$= (\cos\theta + i\ \sin\theta)(\cos\theta + (\sin\theta)$

$= \cos^2\theta + i\ \cos\theta sin\,\theta + cos\theta\ sin\theta + (i)^2\sin^2\theta$

$= \cos^2\theta + 2i\cos\theta sin\,\theta - \sin^2\theta$, But $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$=\sin^2\theta + 2i\ cos\theta\sin\theta - \sin^2\theta$

$=1 - \sin^2\theta + 2i\ cos\theta\sin\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta + 2i\ cos\theta\sin\theta$

But from trigonometric identities, $1 - 2\sin^2\theta = \cos 2\theta$, Substituting this,

$= \cos\ \theta + 2i\cos\theta sin\theta$

But, $2\ cos\theta\sin\theta = \sin 2\theta$ as an identity, substituting back

$= \cos 2\theta + i\sin 2\theta$

Hence, $z^2 = (\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta)$

b) $z^3 = z\ .z^2$

$= (\cos\theta + i\sin\theta)^3 = ((\cos\theta + isin\theta)\ (\cos\theta + \ i\ sin\theta)^2\ )$, Since $x^3 = x.\ x^2$

But $(\cos\theta + i\sin\theta) = \cos 2\theta + i\sin 2\,\theta\ from$ z_2 above. ***Substituting***

$= (\cos\theta cos2\theta + icos\theta sin2\theta + i\ sin\theta cos2\theta - sin\theta sin2\theta \ldots (1)$

But $cos\ (A + B) = cos\ A\ cos\ B - sin\ A\ sin\ B$ is a trigonometric identity

Cos A cos B =cos (A+B) + sin A sin B

$\cos\theta cos2\theta = cos3\theta + sin\theta sin2\theta \ldots (2)$

sin (A+B)= sin A cos B +cos A sin B

cos A cos B = cos A sin B-sin(A+B)

$\cos\theta cos2\theta = cos\theta sin2\theta - sin3\theta \ldots (3)$,

Substituting (2) and (3) in (1)

$= \cos 3\theta + sin\theta sin2\theta + icos\theta sin2\theta + 1sin\theta cos2\theta\theta - sin\theta\theta sin2\theta$

$= \cos 3\theta + icos\theta sin2\theta + isin\theta cos2\theta$

But, sin (A+B)=sin A cos B + sin B cos A

$= \cos 3\theta + i(cos\theta + sin2\theta + \sin(2\theta)cos2\theta)$,

Substituting $\sin(A + B) = \sin A\cos B + \sin B\cos A$

$\therefore cos\theta sin2\theta + sin2\theta cos\theta = sin3\theta$

$= \cos\theta + isin3\theta$

Hence, z^3 **=** $(\cos\theta + isin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$z^3 = \cos 3\theta + i\sin 3\theta$

c) **note**

$$z^2 = (\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta)$$
$$z^3 = \cos 3\theta + i\sin 3\theta$$

Hence, $z^4 = (\cos\theta + isin\theta)^4 = \cos 4x + i\sin 4\theta$

i. You notice that,

$z = \cos\theta + i\ sin\theta$

$z^2 = (\cos\theta + i\ sin\theta)^2 = cos2\theta + i\ sin2\theta$

$z^3 = (\cos\theta + i\ sin\theta)^3 = cos3\theta + i\ sin3\theta$

$z^4 = (\cos\theta + i\ sin\theta)^4 = cos4\theta + i\ sin4\theta$

Exercise 6.3

Solution

1. $\sin 3\theta = 3sinx + 4sin^3x$

De Moivre's Theorem and the roots of complex numbers

Task

Plot the roots of $8^{\frac{1}{3}}$ on an Argand diagram.

What do you notice?

Exercise 6.4

Solutions

1. 4096
2. $z_0 = e^{\frac{5\pi}{8i}}, z_1 = e^{\frac{5\pi}{8}}, = z_2 e^{\frac{9\pi}{8}}, z_3 = e^{\frac{13}{8}}, z_4 = e$

3. $-64\sqrt{3} - 64i$
4. -16
5. $z_2 = 2^{1/5}e^{78i}$

   $z_1 = 2^{1/5}e^{6i}$

   $z_3 = 2^{1/5}e^{150i}$

   $z_4 = 2^{1/5}e^{222i}$

   $z_5 = 2^{1/5}e^{294i}$

6. $2 + 11i$